Matrix models and growth processes: from viscous flows to the quantum Hall effect*

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Abstract

We review the recent developments in the theory of normal, normal self-dual and general complex random matrices. The distribution and correlations of the eigenvalues at large scales are investigated in the large N limit. The $1/N$ expansion of the free energy is also discussed. Our basic tool is a specific Ward identity for correlation functions (the loop equation), which follows from invariance of the partition function under reparametrizations of the complex eigenvalues plane. The method for handling the loop equation requires the technique of boundary value problems in two dimensions and elements of the potential theory. As far as the physical significance of these models is concerned, we discuss, in some detail, the recently revealed applications to diffusion-controlled growth processes (e.g., to the Saffman-Taylor problem) and to the semiclassical behaviour of electronic blobs in the quantum Hall regime.

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1   Introduction

The subject matter of random matrix theory is a matrix whose entries are randomly distributed with some probability density. To put it another way, the theory deals with statistical ensembles of matrices. Given such an ensemble, one is typically interested in the distribution of eigenvalues and correlations between them as size of the matrices, N, tends to infinity. The distribution and correlation laws obtained in this way turn out to be common to objects and systems of very diverse nature.

The area of applications of the random matrix theory in physics (and mathematics) is enormously vast. It ranges from energy levels statistics in nuclei to number theory, from quantum chaos to string theory. Most extensively employed and best-understood are ensembles of hermitian or unitary matrices, with eigenvalues being confined either to the real line or to the unit circle. Their applications to the level statistics in nuclei go back to Wigner's works of early 50-s. For different aspects of random matrix theory, its applications and related topics see e.g. [1]-[4].

In these lectures we consider more general classes of random matrices, with no a priori restrictions to their eigenvalues being imposed. The eigenvalues can be arbitrary complex numbers. Such models are as yet less well understood but they are equally interesting and meaningful. As we shall see, they may exhibit even richer mathematical structures than their Hermitian counterparts. Their physical applications are also many and varied. (A list of the relevant physical problems and corresponding references can be found in, e.g., [5].) The present lectures are based on our recent works [6]-[10] where new applications to diffusion limited growth processes, complex analysis and quantum Hall effect were found.

The progenitor of ensembles of matrices with general complex eigenvalues is the statistical model of complex matrices with the Gaussian weight. It was introduced by Ginibre [11] in 1965. The partition function of this model is

$$Z_N = \int [D\Phi] \exp\left(-\frac{N}{t} \mathrm{tr}\, \Phi^\dagger \Phi \right)$$

Here $[D\Phi] = \prod_{ij} d(\mathcal{R}e\,\Phi_{ij}) d(\mathcal{I}m\,\Phi_{ij})$ is the standard volume element in the space of $N \times N$ matrices with complex entries Φ_{ij} and t is a (real positive) parameter. Along with the Ginibre ensemble and its generalizations we also consider ensembles of normal matrices, i.e., such that Φ commutes with its hermitian conjugate $\Phi^\dagger$, and normal self-dual matrices (the definition follows below in Section 2).

Since one is primarily interested in statistics of eigenvalues, it is natural to express the probability density in terms of complex eigenvalues $z_j = x_i + iy_j$ of the matrix Φ. It appears that the volume element can be represented as

$$[D\Phi] \propto \prod_{i<j} |z_i - z_j|^{2\beta} \prod_i d^2 z_i$$

where $\beta = 1$ for complex and normal matrices and $\beta = 2$ for normal self-dual matrices. If the statistical weight depends on the eigenvalues only, as it is usually assumed, the other parameters of the matrix (often referred to as "angular variables") are irrelevant and can be integrated out giving an overall normalization factor. In this case the original matrix

problem reduces to statistical mechanics of N particles with complex coordinates z_j in the plane. We thus see that even if the matrix entries Φ_{ij} are statistically independent, like in the Ginibre ensemble, the eigenvalues are correlated in a nontrivial way. Specifically, the factor $\prod_{i<j} |z_i - z_j|^{2\beta}$, being equal to the exponentiated Coulomb energy in two dimensions, means an effective "repelling" of eigenvalues. This remark leads to the Dyson logarithmic gas interpretation [12], which treats the matrix ensemble as a two-dimensional "plasma" of eigenvalues in an external electric field.

At $\beta = 1$, there is another important interpretation. Namely, the factor $\prod_{i<j} (z_i - z_j)$ can be thought of as coming from the Slater determinant of one-particle fermionic states. The averaging over matrices then turns into averaging over the ground state of a system of N non-interacting fermions. In the case of the Ginibre ensemble it is the system of N electrons in a uniform magnetic field at the lowest Landau level. In the case of a spin-$\frac{1}{2}$ electron in a non-uniform magnetic field all energy levels split, with the only exception of the lowest one, which remains highly degenerate. We shall see that the normal and complex matrix ensembles with a non-Gaussian statistical weight are equivalent to N polarized electrons in a non-uniform magnetic field confined to the lowest energy level. If the degeneracy of the level equals N, i.e., if the level is completely filled, the system of N electrons behaves as an incompressible quantum Hall droplet [13].

When N becomes large some new features emerge, which require a different language for their adequate description, in much the same way as classical thermodynamics results from statistical mechanics. As $N \to \infty$, the eigenvalues densely fill a domain in the complex plane with the mean density outside it being exponentially small in N. Around the edge of this domain the density steeply drops down. The width of the transition region tends to zero as $N \to \infty$, so that the density profile in the direction normal to the edge looks like a step function. This fact allows one to introduce the *support of eigenvalues* to be the region where the mean density of eigenvalues does not vanish as $N \to \infty$. Typically, it is a bounded domain (or several disconnected domains) in the complex plane. Its shape is determined by the probability density.

For the Ginibre ensemble, the support of eigenvalues is the disk of radius $\sqrt{t}$ with uniform density. It is the counterpart of the celebrated Wigner "semicircular law". For matrix ensembles with non-Gaussian weights the supports are in general not circular and not connected. Throughout these lectures our attention is mostly restricted to the case when the support of eigenvalues is a connected domain. Even in this relatively simpler case, the shape of this domain depends on parameters of the statistical weight in a rather complicated way. As we shall see in Section 4, the problem to find the support of eigenvalues from a given statistical weight is equivalent to the inverse problem of potential theory in two dimensions. In most cases, solutions of the latter are not available in an explicit form.

Nevertheless, the local dynamical law that governs the evolution of the support of eigenvalues under changes of parameters of the statistical weight (like t in the Ginibre ensemble) can be expressed in terms of the exterior Dirichlet boundary value problem. Namely, the edge of the support moves along gradient of a scalar harmonic field in its exterior, with the velocity being proportional to the absolute value of the gradient. Remarkably, this growth law is known to be common to a wide class of diffusion-limited growth processes of which the most popular example is viscous flow in the Hele-Shaw cell (see [14] for a review). The mentioned above equivalence between the normal matrix

ensemble and the quantum Hall droplet suggests that the semiclassical behaviour of electronic droplets in a non-uniform magnetic field follows the same laws as the Hele-Shaw flows do.

This fact allows one to treat the model of normal or complex random matrices as a growth problem. The advantage of this viewpoint is two-fold. First, the hydrodynamic interpretation makes some of the large N matrix model results more illuminating and intuitively accessible. Second and most important, the matrix model perspective may help to suggest new approaches to the long-standing growth problems. In this respect, of special interest is the identification of finite time singularities in some exact solutions to the Hele-Shaw flows with critical points of the normal and complex matrix models.

At last, a few words about the organization of the lectures. The material that follows can be divided into three parts. The first one (Section 2) can be regarded as a continuation of the introduction. We define the main matrix ensembles to be considered and give their physical interpretations. The second part (Section 3) contains exact results valid at any finite N. We outline the integrable structure of the normal and complex models at $\beta = 1$ (the Hirota relations for the partition function, the orthogonal polynomials technique and the Lax representation). In addition, we derive the exact relation between correlation functions of the eigenvalue densities (referred to as the loop equation) which holds for arbitrary values of β. In the third part (Sections 4 and 5) we examine the large N limit of the models of random matrices with complex eigenvalues and discuss the applications to the growth processes and to the semiclassical electronic droplets in magnetic field. The Appendices contain technical details of some proofs and calculations.

2   Some ensembles of random matrices with complex eigenvalues

We consider square random matrices Φ of size N with complex entries Φ_{ij} subject to certain constraints depending on the particular ensemble. Some ensembles of random matrices are listed in the following table:

Ensemble	Notation	Condition	Dimension
Hermitian	$\mathcal{H}$	$\Phi^\dagger = \Phi$	N^2
Unitary	$\mathcal{U}$	$\Phi^\dagger \Phi = 1$	N^2
Normal	$\mathcal{N}$	$[\Phi^\dagger, \Phi] = 0$	$N^2 + N$
Normal self-dual	$\mathcal{N}^0$	$[\Phi^\dagger, \Phi] = 0$ Φ self-dual	$\frac{1}{2}N^2 + N$
Complex	$\mathcal{C}$	none	$2N^2$

The first two matrix ensembles, $\mathcal{H}$ and $\mathcal{U}$, are the most popular ones. They are given here just for comparison. Eigenvalues of matrices from $\mathcal{H}$ and $\mathcal{U}$ are confined to the real axis and to the unit circle respectively. The last three ensembles (which are the main subject of these lectures) do not imply any a priori restrictions on eigenvalues of the matrices. Normal matrices are defined by the constraint that they commute with their adjoint. The ensemble $\mathcal{N}^0$ is defined for N even only. The meaning of the condition "Φ is self-dual" is explained below in this section. By dimension of the ensemble we mean the *real* dimension of the matrix variety.

Throughout this paper we consider the probability densities of the form $P(\Phi) \propto e^{\mathrm{tr} W(\Phi)}$, where the function $W(\Phi)$ (often called the potential of the matrix model) is a matrix-valued function of Φ and $\Phi^\dagger$ such that $(W(\Phi))^\dagger = W(\Phi)$. This form is similar to the one usually employed in Hermitian and unitary ensembles. The partition function is defined as the integral over matrices from one or another ensemble:

$$Z_N = \int [D\Phi] e^{\mathrm{tr}\, W(\Phi)} \tag{2.1}$$

Summing over N with a suitable weight, one may also define the grand canonical ensembles corresponding to (2.1) but we do not pursue this possibility here.

We need to specify the integration measure $[D\Phi]$ and the potential $W(\Phi)$. Given the measure and the potential, one is usually interested in the distribution and correlations of the eigenvalues. In general, they can be distributed on the real line for $\mathcal{H}$, on the unit circle for $\mathcal{U}$ and on the whole complex plane for $\mathcal{N}$, $\mathcal{N}^0$ and $\mathcal{C}$.

2.1 Integration measures

The integration measure has the most simple form for the ensemble of general complex matrices:

$$[D\Phi] = \prod_{i,j=1}^{N} d(\mathcal{R}e\, \Phi_{ij})\, d(\mathcal{I}m\, \Phi_{ij})$$

This measure is additively invariant and multiplicatively covariant, i.e. for any fixed (nondegenerate) matrix $A \in \mathcal{C}$ we have the properties $[D(\Phi + A)] = [D\Phi]$ and $[D(\Phi A)] = [D(A\Phi)] = |\det A|^{2N}[D\Phi]$. The first one is obvious, to prove the second one is an easy exercise. It is clear that the measure is invariant under transformations of the form $\Phi \to U^\dagger \Phi U$ with a unitary matrix U ("rotations" in the matrix space).

The measure for $\mathcal{N}$ is induced by the standard flat metric in $\mathcal{C}$,

$$||\delta\Phi||^2 = \mathrm{tr}\,(\delta\Phi\delta\Phi^\dagger) = \sum_{ij} |\delta\Phi_{ij}|^2$$

via the embedding $\mathcal{N} \subset \mathcal{C}$. Here $\mathcal{N}$ is regarded as a hypersurface in $\mathcal{C}$ defined by the quadratic relations $\Phi\Phi^\dagger = \Phi^\dagger\Phi$. The measure for the ensemble $\mathcal{N}^0$ is defined in a similar way.

As usual in matrix models, we would like to integrate out the "angular" variables and to express the integration measure through eigenvalues of the matrices.

The measure for $\mathcal{N}$ through eigenvalues [1, 15]. We derive the explicit representation of the measure in terms of eigenvalues in three steps:

1. Introduce coordinates in $\mathcal{N} \subset \mathcal{C}$.

2. Compute the inherited metric on $\mathcal{N}$ in these coordinates: $||\delta\Phi||^2 = g_{\alpha\beta}d\xi^\alpha d\xi^\beta$.

3. Compute the volume element $[D\Phi] = \sqrt{|\det g_{\alpha\beta}|}\prod_\alpha d\xi^\alpha$.

Step 1: Coordinates in $\mathcal{N}$. For any matrix Φ, the matrices $H_1 = \frac{1}{2}(\Phi + \Phi^\dagger)$, $H_2 = \frac{1}{2i}(\Phi - \Phi^\dagger)$ are Hermitian. The condition $[\Phi, \Phi^\dagger] = 0$ is equivalent to $[H_1, H_2] = 0$. Thus $H_{1,2}$ can be simultaneously diagonalized by a unitary matrix U:

$$H_1 = UXU^\dagger, \quad X = \mathrm{diag}\,\{x_1, \ldots, x_N\}$$
$$H_2 = UYU^\dagger, \quad Y = \mathrm{diag}\,\{y_1, \ldots, y_N\}$$

Introduce the diagonal matrices $Z = X + iY$, $\bar{Z} = X - iY$ with diagonal elements $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$ respectively. Note that z_j are eigenvalues of Φ. Therefore, any $\Phi \in \mathcal{N}$ can be represented as

$$\Phi = UZU^\dagger$$

where U is a unitary matrix and Z is the diagonal matrix with eigenvalues of Φ on the diagonal. In fact normal matrices can be equivalently defined by the property of being the most general matrices that can be diagonalized by a unitary transformation. The matrix U is defined up to multiplication by a diagonal unitary matrix from the right: $U \to U\,U_{\mathrm{diag}}$. The dimension of $\mathcal{N}$ is thus

$$\dim\,(\mathcal{N}) = \dim\,(\mathcal{U}) - \dim\,(\mathcal{U}_{\mathrm{diag}}) + \dim\,(\mathcal{C}_{\mathrm{diag}}) = N^2 - N + 2N = N^2 + N$$

Let us make a remark that the naive counting of the number of constraints in the condition $[\Phi, \Phi^\dagger] = 0$ leads to a wrong result for the dimension of $\mathcal{N}$. On the first glance, this condition gives $N^2 - 1$ independent constraints. Indeed, set $H = [\Phi, \Phi^\dagger]$. Then the conditions $H_{lk} = 0$ for $l < k$ give $N(N-1)$ real constraints and the conditions $H_{kk} = 0$ give $N - 1$ real constraints (because $\mathrm{tr}\,H = 0$ identically), in total $N^2 - 1$ constraints. We thus observe that $\dim\,(\mathcal{N}) \neq \dim\,(\mathcal{C}) - (N^2 - 1)$. Therefore, there are only $N^2 - N$ independent constraints among the $N^2 - 1$ equations $[\Phi, \Phi^\dagger] = 0$. This fact can be easily illustrated by the example of 2×2 matrices.

Step 2: The induced metric. Since $\Phi = UZU^\dagger$, the variation is $\delta\Phi = U(\delta u \cdot Z + \delta Z + Z \cdot \delta u^\dagger)U^\dagger$, where $\delta u^\dagger = U\delta U^\dagger = -\delta u^\dagger$. Therefore,

$$||\delta\Phi||^2 = \mathrm{tr}\,(\delta\Phi\delta\Phi^\dagger) = \mathrm{tr}\,(\delta Z\delta\bar{Z}) + 2\,\mathrm{tr}\,(\delta uZ\delta u\bar{Z} - (\delta u)^2 Z\bar{Z})$$

$$= \sum_{j=1}^N |\delta z_j|^2 + 2\sum_{j<k}^N |z_j - z_k|^2\,|\delta u_{jk}|^2$$

(Note that δu_{jj} do not enter.) This is the square of the line element $||\delta\Phi||^2 = g_{\alpha\beta}\delta\xi^\alpha\delta\xi^\beta$.

Step 3. The volume element. We see that the metric $g_{\alpha\beta}$ is diagonal in the coordinates $\mathcal{R}e(\delta z_j)$, $\mathcal{I}m(\delta z_j)$, $\mathcal{R}e(\delta u_{jk})$, $\mathcal{I}m(\delta u_{jk})$ with $1 \leq j < k \leq N$, so the determinant of the diagonal matrix $g_{\alpha\beta}$ is easily calculated to be $|\det g_{\alpha\beta}| = 2^{N^2-N} \prod_{j<k}^{N} |z_i - z_k|^4$. Therefore,

$$[D\Phi] \propto [DU]' \, |\Delta_N(z_1, \ldots, z_N)|^2 \prod_{j=1}^{N} d^2 z_j \tag{2.2}$$

where $d^2 z \equiv dxdy$ is the flat measure in the complex plane, $[DU]' = [DU]/[DU_{\text{diag}}]$ is the invariant measure on $\mathcal{U}/\mathcal{U}_{\text{diag}}$, and

$$\Delta_N(z_1, \ldots, z_N) = \prod_{\substack{j>k}}^{N} (z_j - z_k) = \det_{N \times N} (z_j^{k-1}) \tag{2.3}$$

is the Vandermonde determinant.

Similarly to the ensemble $\mathcal{H}$ of Hermitian matrices, the measure (2.2) contains the squared modulus of the Vandermonde determinant. The difference is that the eigenvalues are complex numbers. The statistical model of normal random matrices was studied in [15, 16].

Normal self-dual matrices. Let Γ be the matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}, \quad \Gamma^2 = -1$$

(all other entries are zero). A complex matrix Φ is called *self-dual* if $\Gamma \Phi^{\mathrm{T}} \Gamma = -\Phi$ (the superscript T means transposition). The size of a self-dual matrix is thus an even number. It can be shown that eigenvalues of self-dual matrices always come in pairs: the diagonal form of Φ is

$$Z = \text{diag} \{z_1, z_1, \, z_2, z_2, \ldots, z_N, z_N\}$$

As is easy to verify, the condition that Φ is self-dual is equivalent to the condition that the matrix $\Gamma\Phi$ is anti-symmetric.

Normal self-dual matrices are parameterized as $\Phi = UZU^\dagger$ with Z as above, where U is unitary and symplectic: $U^\dagger U = 1$, $U^{\mathrm{T}}\Gamma U = \Gamma$. In other words, U belongs to the maximal compact subgroup in the complex group of symplectic matrices $\mathrm{Sp}(N)$. The latter is known to have real dimension $4N^2 + 2N$, with the dimension of the maximal compact subgroup being twice less. Therefore, similarly to the calculation for normal matrices, $\dim(\mathcal{N}^0) = 2N^2 + N - N + 2N = 2N^2 + 2N$ (the value given in the table above corresponds to N replaced by $N/2$). The integration measure appears to be

$$[D\Phi] \propto [DU]' \, |\Delta_N(z_1, \ldots, z_N)|^4 \prod_{j=1}^{N} d^2 z_j \tag{2.4}$$

(for $2N \times 2N$ matrices). Note that the module of the Vandermonde determinant enters in the fourth degree. The statistical model of normal self-dual matrices was discussed in [17].

The measure for $\mathcal{C}$ through eigenvalues. A complex matrix Φ with eigenvalues $z_1, \ldots, z_N$ can be decomposed as

$$\Phi = U(Z + R)U^\dagger$$

where $Z = \mathrm{diag}\,\{z_1, \ldots, z_N\}$ is diagonal, U is unitary, and R is strictly upper triangular, i.e., $R_{ij} = 0$ if $i \geq j$. These matrices are defined up to a "gauge transformation": $U \to U\,U_{\mathrm{diag}}$, $R \to U_{\mathrm{diag}}^\dagger\,R\,U_{\mathrm{diag}}$. It is not so easy to see that the measure factorizes. This requires some work, of which the key step is a specific ordering of the independent variables. The final result is:

$$[D\Phi] \propto [DU]' \left(\prod_{k<l} d^2 R_{kl} \right) |\Delta_N(z_i)|^2 \prod_{j=1}^{N} d^2 z_j \tag{2.5}$$

The details can be found in the Mehta book [1].

2.2   Potentials

For the ensembles $\mathcal{N}$, $\mathcal{N}^0$ the "angular variables" (parameters of the unitary matrix U) always decouple after taking the trace $\mathrm{tr}\,W(\Phi) = \sum_j W(z_j)$, so the potential W can be a function of Φ, $\Phi^\dagger$ of a general form $W(\Phi) = \sum a_{nm}\Phi^n(\Phi^\dagger)^m$. Two important particular cases arise if the potential is:

- Axially symmetric, $W(\Phi) = W_0(\Phi\Phi^\dagger)$. In this case the N-fold integral essentially reduces to ordinary ones, and so some basic results become available in a quite explicit form.

- Harmonic on the background of $\Phi\Phi^\dagger$, i.e., $W(\Phi) = -\Phi\Phi^\dagger + V(\Phi) + \bar{V}(\Phi^\dagger)$. In what follows, we call it *quasiharmonic*. Here $V(z)$ is an *analytic* function of z in some domain containing the origin and $\bar{V}(z) = \overline{V(\bar{z})}$. In terms of the eigenvalues, the quasiharmonic potential is

$$W(z) = -|z|^2 + V(z) + \overline{V(z)} \tag{2.6}$$

  This case is particularly important for applications. The normal matrix model with quasiharmonic potentials bears some formal similarities with the model of two coupled Hermitian matrices [18] and the matrix quantum mechanics in the singlet sector [19].

The partition function reduces to

$$Z_N = \int |\Delta_N(z_i)|^{2\beta} \prod_{j=1}^{N} e^{W(z_j)} d^2 z_j \tag{2.7}$$

where $\beta = 1$ for $\mathcal{N}$ and $\beta = 2$ for $\mathcal{N}^0$. From now on this formula is taken as the definition of the partition function. Comparing to (2.1), we redefine $W \to W/\beta$ and ignore a possible N-dependent normalization factor. One may also consider this integral for arbitrary values of β.

The choice of the potential for the ensemble $\mathcal{C}$ is more restricted. For a general potential, the matrix U in $\Phi = U(Z+R)U^\dagger$ still decouples but R does not. The problem becomes too complicated. An important particular case, when R nevertheless decouples is the quasiharmonic potential (2.6). Indeed, $\mathrm{tr}\,(\Phi\Phi^\dagger) = \mathrm{tr}\,(Z\bar{Z}) + \mathrm{tr}\,(RR^\dagger)$, $\mathrm{tr}\,(\Phi^n) = \mathrm{tr}\,(Z+R)^n = \mathrm{tr}\,Z^n$, and so

$$\int_{\mathcal{C}} [D\Phi] e^{\mathrm{tr}\,W(\Phi)} = C_N \int |\Delta(z_i)|^2 \prod_k e^{W(z_k)} d^2 z_k \qquad (2.8)$$

where C_N is an N-dependent normalization factor proportional to the gaussian integral $\int [DR] e^{-\mathrm{tr}\,(RR^\dagger)}$.

As an example, let us consider the quadratic potential:

$$W(z) = -\sigma|z|^2 + 2\mathcal{R}e\,(t_1 z + t_2 z^2), \quad \sigma > 0$$

The ensemble $\mathcal{C}$ ($\beta = 1$) with this potential is known as *the Ginibre-Girko ensemble* [11, 20]. In this case the partition function (2.7) can be calculated exactly [21]:

$$Z_N = Z_N^{(0)} (\sigma^2 - 4|t_2|^2)^{-N^2/2} \exp\left(N\,\frac{t_1^2 \bar{t}_2 + \bar{t}_1^2 t_2 + \sigma|t_1|^2}{\sigma^2 - 4|t_2|^2} \right)$$

where

$$Z_N^{(0)} = \sigma^{(N^2-N)/2} \pi^N \prod_{k=1}^{N} k!$$

To the best of our knowledge, there are no exact results for 2D integrals of this type with $|\Delta(z_i)|^{2\beta}$ for other values of β, even for the pure Gaussian weight.

Coming back to the general case, we note that some integrals considered above may diverge. In the most important case of quasiharmonic potential, the integral

$$\int e^{-|z|^2 + V(z) + \overline{V(z)}} d^2 z$$

converges for potentials $V(z) = \alpha z^2 + \beta z + \sum_i \mu_i \log(z - a_i)$ with $|\alpha| < \frac{1}{2}$ and $\mu_j > -1$ but it always diverges if $V(z)$ is a polynomial of degree ≥ 3. As usual in matrix models, really interesting science begins when integrals diverge! Let us say a few words about how one should understand divergent integrals. The conventional viewpoint is to treat all cubic and higher degree terms in the potential as a small perturbation. The integral is then regarded as a perturbative series for a theory which is believed to be well-defined on the nonperturbative level. The nonperturbative definition can be achieved either by introducing an ad hoc cutoff or via more sophisticated methods in the spirit of the Marinari-Parisi approach [22] (the "stochastic stabilization"). As far as the large N limit is concerned, the integral for the partition function can be *defined* through the expansion around the saddle point. This gives the $1/N$-expansion

$$\log Z_N \sim \sum_h N^{-h} F^{(h)} \qquad \text{as } N \to \infty$$

Even if the integral for Z_N diverges, each term of the $1/N$ expansion is often well-defined.

2.3 Physical interpretations

The ensembles of random matrices appear to be mathematically equivalent to some important model systems of statistical and quantum mechanics. They are:

- The 2D Coulomb plasma (any β)
- Non-interacting fermions ($\beta = 1$)
- Electrons in magnetic field ($\beta = 1$)

The equivalence holds for any finite N. The first two interpretations are standard and well known. The third one can be regarded as a specification of the second one for models with eigenvalues distributed over the whole complex plane. Remarkably, in this very case the noninteracting fermions picture (which is rather formal for $\mathcal{H}$ and $\mathcal{U}$) acquires a very interesting physical content related to the quantum Hall effect.

The Dyson gas picture. This interpretation, first suggested by Dyson [12] for the unitary, symplectic and orthogonal matrix ensembles, relies on rewriting $|\Delta_N(z_i)|^{2\beta}$ as $\exp\left(\beta \sum_{i \neq j} \log |z_i - z_j|\right)$. Clearly, the integral (2.7) looks then exactly as the partition function of the 2D Coulomb plasma (often called the Dyson gas) at "temperature" $1/\beta$, in the external electric field:

$$Z_N = \int e^{-\beta E(z_1, \ldots, z_N)} \prod d^2 z_j \qquad (2.9)$$

The eigenvalues play the role of the 2D Coulomb charges. The energy is

$$E = -\sum_{i<j} \log |z_i - z_j|^2 - \beta^{-1} \sum_j W(z_j) \qquad (2.10)$$

The first sum is the Coulomb interaction energy, the second one is the energy due to the external field. For the ensembles $\mathcal{H}$ and $\mathcal{U}$ the charges are confined to dimension 1 (the real line or the unit circle) but interact as 2D Coulomb charges. So, the Dyson gas picture for the ensembles $\mathcal{N}$, $\mathcal{N}^0$ and $\mathcal{C}$ looks even more natural. The Dyson gas interpretation becomes especially helpful in the large N limit, where it allows one to apply thermodynamical arguments.

Non-interacting fermions ($\beta = 1$). Given any system of polynomials of the form $P_n(z) = z^n +$ lower degrees, the Vandermonde determinant can be written as $\Delta_N(z_i) = \det(z_j^{k-1}) = \det(P_{k-1}(z_j))$. Let us rewrite the statistical weight as

$$|\Delta_N(z_i)|^2 \prod_{j=1}^{N} e^{W(z_j)} = |\Psi(z_1, \ldots, z_N)|^2$$

where $\Psi(z_1, \ldots, z_N) = \det_{N \times N}\left(P_{k-1}(z_j) e^{W(z_j)}\right)$ is the (unnormalized) wave function of N non-interacting fermions (the Slater determinant), with one-particle wave functions being $\psi_k(z) = P_{k-1}(z) e^{W(z)}$. The partition function is the normalization integral:

$$Z_N = \int |\Psi|^2 \prod_i d^2 z_i$$

For the ensembles $\mathcal{N}$ and $\mathcal{C}$ (with quasiharmonic potential) the wave function Ψ has a direct physical meaning as a wave function of 2D electrons in magnetic field.

Electrons in the plane in magnetic field. Consider a charged particle with spin $\frac{1}{2}$ (electron) moving in the plane in a strong (not necessarily uniform) magnetic field $B = B(x, y)$ orthogonal to the plane. The Pauli Hamiltonian reads

$$\hat{H} = \frac{1}{2m}\left((i\hbar\nabla + \vec{A})^2 - \hbar\sigma_3 B\right)$$

Here m is mass of the particle, $\sigma_3 = \mathrm{diag}\,(1, -1)$ is the Pauli matrix, $B = \partial_x A_y - \partial_y A_x$. In 2D, the complex notation is convenient: $A = A_x - iA_y$, $\bar{A} = A_x + iA_y$, so that $B = i(\bar{\partial}A - \partial\bar{A})$.

For a uniform magnetic field, $B = B_0$, one can choose the gauge $A = \frac{B_0}{2i}\bar{z}$. Solving the Schrodinger equation $\hat{H}\psi = E\psi$, one gets Landau levels:

$$E_n = \frac{\hbar B_0}{m}\left(n + \frac{1}{2} - s\right), \quad n = 0, 1, 2, \ldots, \quad s = \pm\frac{1}{2}$$

The gap between the levels is proportional to B_0/m. Each level is highly degenerate. The wave functions at the level $E = 0$ are

$$\psi_n = z^n \exp\left(-\frac{B_0}{4\hbar}|z|^2\right)$$

We note in passing that the system of fermions in the magnetic field at the lowest energy level admits a collective field theory description which was discussed in the literature in different contexts (see e.g. [13, 23, 24]).

Let us turn to the problem with a nonuniform magnetic field. Choose the gauge $A = i\,\partial W$ with a real-valued function W, then $B = -2\,\partial\bar{\partial}W$ and $\mathrm{div}\,\vec{A} = \bar{\partial}A + \partial\bar{A} = 0$. Therefore,

$$(i\hbar\nabla + \vec{A})^2 = -4\hbar^2\partial\bar{\partial} + 2i\hbar((\bar{\partial}W)\partial + (\partial W)\bar{\partial}) + |\partial W|^2$$

$$= (2\hbar\partial + \partial W)(-2\hbar\bar{\partial} + \bar{\partial}W) - 2\hbar\,\partial\bar{\partial}W$$

The Hamiltonian can be represented as the 2×2 matrix

$$\hat{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

where $2mH_\pm = (i\hbar\nabla + \vec{A})^2 \pm 2\hbar\,\partial\bar{\partial}W$. In general, the spectral problem for this Hamiltonian does not admit an explicit solution. However, the level $E = 0$ is very special. Note that H_+ factorizes: $H_+ = (2\hbar\partial + \partial W)(-2\hbar\bar{\partial} + \bar{\partial}W)$, so exact wave functions at the level $E = 0$ can be found by solving the *first order* equation

$$H_+\psi = (2\hbar\bar{\partial} - \bar{\partial}W)\psi = 0$$

The general solution is

$$\psi(z) = P(z)\exp\left(\frac{1}{2\hbar}W(z)\right) \tag{2.11}$$

$E = 0$
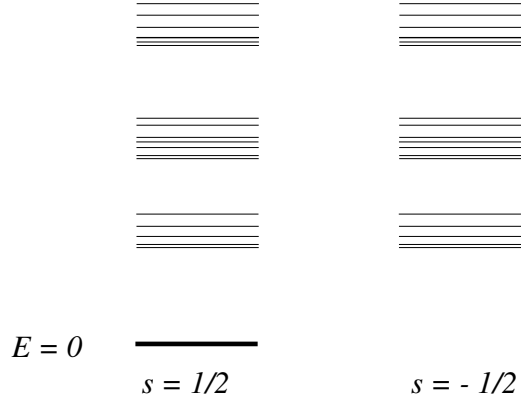
$s = 1/2$ $s = -1/2$

Figure 1: *The energy levels of a spin-$\frac{1}{2}$ electron in a non-uniform magnetic field (schematically). The lowest level $E = 0$ remains highly degenerate.*

where $P(z)$ is an arbitrary *holomorphic* polynomial. The zero energy level remains to be highly degenerate even in the nonuniform magnetic field (Fig. 1). This fact was first observed in [25].

To find degeneracy of the level, we solve the Poisson equation $\Delta W = -2B$,

$$W(z) = -\frac{1}{\pi} \int \log|z - \zeta| B(\zeta) d^2\zeta \qquad (2.12)$$

and observe that $W(z)$ tends to $-\frac{\phi}{\pi} \log|z|$ as $|z| \to \infty$, where $\phi = \int B\, d^2z$ is the total magnetic flux. If $P(z)$ is of degree n, the asymptotics of ψ for $|z|$ large is

$$\psi = P(z)\, e^{\frac{1}{2\hbar}W(z)} \to z^n\, |z|^{-\phi/\phi_0}$$

where $\phi_0 = 2\pi\hbar$ is the flux quantum. We require the wave functions to be normalizable, i.e, $\int |\psi|^2 d^2z < \infty$ that means $n < \phi/\phi_0 - 1$. Therefore,

$$n_{\max} = [\phi/\phi_0] - 1$$

$([\ldots]$ is the integer part), and the degeneracy is equal to the number of flux quanta in the total flux:

$$N = [\phi/\phi_0]$$

If the Coulomb forces can be ignored, the wave function of N electrons in the plane in the magnetic field at the lowest energy level is constructed as the $N \times N$ Slater determinant of the functions of the type (2.11) with polynomials of different degrees.

The situation when the lowest energy level $E = 0$ is completely filled, i.e., $N = n_{\max}$, is the (integer) quantum Hall (QH) regime. The notion of the QH droplet [13] implies that the electronic liquid is incompressible, i.e., all states at the lowest energy level are occupied. We come to the following conclusion: the QH droplet consisting of N electrons (in general, in a non-uniform magnetic field) is equivalent to the ensemble of normal $N \times N$ matrices.

3 Exact results at finite N

3.1 Correlation functions: general relations

The main objects to be determined in random matrix models are correlation functions. In general, they are mean values of scalar-valued functions of matrices. The mean value of such a function $A(\Phi)$ of the matrix Φ is defined, with the help of the statistical weight, in the usual way:

$$\langle A \rangle = \frac{\int [D\Phi] A(\Phi) e^{\operatorname{tr} W(\Phi)}}{\int [D\Phi] e^{\operatorname{tr} W(\Phi)}}$$

We shall consider functions that depend on eigenvalues only – for example, traces of matrices. Correspondingly, typical correlators which we are going to study are mean values of products of traces: $\langle \operatorname{tr} f(\Phi) \rangle$, $\langle \operatorname{tr} f_1(\Phi) \operatorname{tr} f_2(\Phi) \rangle$ and so on. Clearly, they are represented as integrals over eigenvalues. For example,

$$\langle \operatorname{tr} f(\Phi) \rangle = \frac{N \int |\Delta_N(z_i)|^{2\beta} f(z_1) \prod_{j=1}^{N} e^{W(z_j)} d^2 z_j}{\int |\Delta_N(z_i)|^{2\beta} \prod_{j=1}^{N} e^{W(z_j)} d^2 z_j}$$

Here, $f(\Phi) = f(\Phi, \Phi^\dagger)$ is any function of Φ, $\Phi^\dagger$ which is regarded as the function $f(z_i) = f(z_i, \bar{z}_i)$ of the complex argument z_i (and $\bar{z}_i$) in the r.h.s. A particularly important example is the density function defined as

$$\rho(z) = \sum_j \delta(z - z_j) = \operatorname{tr} \delta(z - \Phi) \tag{3.1}$$

where $\delta(z)$ is the two dimensional δ-function. As it immediately follows from the definition, any correlator of traces is expressed through correlators of ρ:

$$\langle \operatorname{tr} f_1(\Phi) \dots \operatorname{tr} f_n(\Phi) \rangle = \int \langle \rho(z_1) \dots \rho(z_n) \rangle f_1(z_1) \dots f_n(z_n) \prod_{j=1}^{n} d^2 z_j \tag{3.2}$$

Instead of correlations of density it is often convenient to consider correlations of the field

$$\varphi(z) = -\beta \sum_j \log |z - z_j|^2 = -\beta \log |\det(z - \Phi)|^2 \tag{3.3}$$

from which the correlations of density can be found by means of the relation

$$4\pi\beta\rho(z) = -\Delta\varphi(z) \tag{3.4}$$

Clearly, φ is the 2D Coulomb potential created by the eigenvalues (charges). As it directly follows from the definitions,

$$\langle \rho(z) \rangle_N = N \frac{Z_{N-1}}{Z_N} \langle e^{W(z) - \varphi(z)} \rangle_{N-1}$$

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where $\langle\ldots\rangle_N$ means the expectation value in the ensemble of $N \times N$ matrices.

Handling with multipoint correlation functions, it is customary to pass to their *connected parts*. For example, in the case of 2-point functions, the connected correlation function is defined as

$$\langle \rho(z_1)\rho(z_2)\rangle_c \equiv \langle \rho(z_1)\rho(z_2)\rangle - \langle \rho(z_1)\rangle \langle \rho(z_2)\rangle$$

The connected multi-trace correlators are expressed through the connected density correlators by the same formula (3.2) with $\langle \rho(z_1)\ldots\rho(z_n)\rangle_c$ in the r.h.s. The connected part of the $(n+1)$-point density correlation function is given by the linear response of the n-point one to a small variation of the potential. More precisely, the following variational formulas hold true:

$$\langle \rho(z)\rangle = \frac{\delta \log Z_N}{\delta W(z)}, \quad \langle \rho(z_1)\rho(z_2)\rangle_c = \frac{\delta \langle \rho(z_1)\rangle}{\delta W(z_2)} = \frac{\delta^2 \log Z_N}{\delta W(z_1)\delta W(z_2)} \tag{3.5}$$

Connected multi-point correlators are higher variational derivatives of $\log Z_N$. These formulas follow from the fact that variation of the partition function over a general potential W inserts $\sum_i \delta(z - z_i)$ into the integral. Let us stress that these formulas are exact for any finite N.

For a later use, we mention the formula

$$\left\langle e^{\operatorname{tr} f(\Phi)} \right\rangle = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k!} \left\langle (\operatorname{tr} f(\Phi))^k \right\rangle_c \right) \tag{3.6}$$

which immediately follows from the expansion

$$\log \left\langle e^{\operatorname{tr} f} \right\rangle = \log Z_N(W + f) - \log Z_N(W)$$

$$= \int \frac{\delta \log Z_N}{\delta W(\zeta)} f(\zeta)d^2\zeta + \frac{1}{2!} \int \frac{\delta^2 \log Z_N}{\delta W(\zeta)\delta W(\zeta')} f(\zeta)f(\zeta')d^2\zeta d^2\zeta' + \ldots$$

3.2 Integrable structure of the $\mathcal{N}$ and $\mathcal{C}$ ensembles ($\beta = 1$)

The partition function (2.7) for $\beta = 1$,

$$Z_N = \int |\Delta_N(z_i)|^2 \prod_{j=1}^N e^{W(z_j)} d^2 z_j$$

regarded as a function of N and Taylor coefficients of the potential W, has remarkable properties, which we briefly review below.

Determinant representation. The following simple but important determinant representation holds true:

$$Z_N = N! \det_{N \times N}(C_{ij}), \quad 1 \le i,j \le N \tag{3.7}$$

where

$$C_{ij} = \int z^{i-1}\bar{z}^{j-1}e^{W(z)}d^2z \qquad (3.8)$$

is the "matrix of moments". The proof is almost a repetition of the corresponding proof for the hermitian model. Complexity of eigenvalues does not cause any difficulties. For completeness, the detailed proof is given in the Appendix.

Hirota relations. Let us apply the determinant formula to $\langle\det(\lambda - \Phi)\rangle$:

$$\langle\det(\lambda - \Phi)\rangle = \frac{1}{Z_N}\int |\Delta_N|^2 \prod_j (\lambda - z_j)e^{W(z_j)}d^2z_j = \frac{1}{Z_N}\det\left[z^{i-1}\bar{z}^{j-1}(\lambda-z)e^W d^2z\right]$$

Comparing this with the determinant representation of Z_N, we see that $\langle\det(\lambda - \Phi)\rangle = \lambda^N Z_N^{-1}\det\left[C_{ij} - \lambda^{-1}C_{i+1,j}\right]$. Taking appropriate linear combinations of the lines, one can reduce this determinant to the determinant of a matrix which differs from C_{ij} only in the last line. Similarly, in the determinant representation of $\langle\det(\lambda_1 - \Phi)\det(\lambda_2 - \Phi)\rangle$ only two last lines change. Now, some standard identities for determinants lead to the following relations:

$$(\lambda_1-\lambda_2)\langle\det(\lambda_1-\Phi)\det(\lambda_2-\Phi)\rangle\langle\det(\lambda_3-\Phi)\rangle + \text{cyclic perm-s of } (123) = 0 \qquad (3.9)$$

$$\langle|\det(\lambda-\Phi)|^2\rangle_N - |\langle\det(\lambda-\Phi)\rangle_N|^2 = \frac{N}{N+1}\frac{Z_{N+1}Z_{N-1}}{Z_N^2}\langle|\det(\lambda-\Phi)|^2\rangle_{N-1} \qquad (3.10)$$

Write

$$\langle\det(\lambda-\Phi)\rangle_N = \lambda^N \frac{Z_N(W + [\lambda])}{Z_N(W)}$$

where

$$W + [\lambda] \equiv W(z) + \log\left(1 - \frac{z}{\lambda}\right)$$

is the potential modified by the (complex and multi-valued) term $\log\left(1 - \frac{z}{\lambda}\right)$. (Since this term is always under the exp-function, there is no ambiguity in the choice of its branch.) In this notation, the above identities for determinants acquire the form of the *Hirota bilinear equations* [26]:

$$(\lambda_1-\lambda_2) Z_N(W+[\lambda_1]+[\lambda_2]) Z_N(W+[\lambda_3]) + \text{cyclic perm-s of } 1,2,3 = 0 \qquad (3.11)$$

$$Z_N(W)Z_N(W+[\lambda]+[\bar{\lambda}]) - Z_N(W+[\lambda])Z_N(W+[\bar{\lambda}])$$

$$= \frac{N}{N+1}|\lambda|^{-2}Z_{N+1}(W)\,Z_{N-1}(W+[\lambda]+[\bar{\lambda}]) \qquad (3.12)$$

Let us parameterize the potential as

$$W(z) = W^{(0)}(z) + \sum_k (t_k z^k + \bar{t}_k\bar{z}^k)$$

then these equations state that $Z_N/N!$, as a function of t_k, $\bar{t}_k$, is the *tau-function* of the 2D Toda lattice hierarchy. The transformation $W \to W + [\lambda]$ is equivalent to the change of variables $t_k \to t_k - \frac{1}{k}\lambda^{-k}$ which is known in the literature as the Miwa transformation.

Orthogonal polynomials and the kernel function. The orthogonal polynomials technique is useful not only for hermitian and unitary matrix ensembles but for the normal and complex ensembles as well. The orthogonal polynomials are introduced as mean values of the characteristic polynomials of the random matrices Φ:

$$P_n(\lambda) = \langle \det(\lambda - \Phi) \rangle_n \tag{3.13}$$

Clearly, P_n are polynomials in λ of the form $P_n(\lambda) = \lambda^n +$ lower degrees.

The main property of the polynomials introduced is their *orthogonality* in the complex plane:

$$\int P_n(z) \overline{P_m(z)} e^{W(z)} d^2z = h_n \delta_{mn} \tag{3.14}$$

The square of the norm $h_n = ||P_n||^2$ is connected with the partition function as

$$h_n = \frac{1}{n+1} \frac{Z_{n+1}}{Z_n}, \qquad Z_N = N! \prod_{n=0}^{N-1} h_n$$

Again, the proof is completely parallel to the corresponding proof in the hermitian models. See Appendix for details.

The functions

$$\psi_n(z) = \frac{1}{\sqrt{h_{n-1}}} P_{n-1}(z) e^{W(z)/2}$$

are *orthonormal*:

$$\int \psi_n(z) \overline{\psi_m(z)} d^2z = \delta_{mn} \tag{3.15}$$

These ψ_n's are "one-particle wave functions" of electrons in the magnetic field. The N-particle wave function is $\Psi_N(z_1, \ldots, z_N) \sim \det[\psi_j(z_k)]$. The joint probability to find "particles" at $z_1, \ldots, z_N$ is

$$|\Psi_N(z_1, \ldots, z_N)|^2 = \frac{1}{N!} |\det[\psi_j(z_k)]|^2$$

Since $|\det M|^2 = \det(MM^\dagger)$, we can write: $|\det[\psi_j(z_k)]|^2 = \det\left(\sum_{n=1}^{N} \psi_n(z_j) \overline{\psi_n(z_k)}\right)$. The expression under the determinant is called the *kernel function*:

$$K_N(z, \bar{w}) = \sum_{n=1}^{N} \psi_n(z) \overline{\psi_n(w)} \tag{3.16}$$

The main properties of the kernel function are:

- Hermiticity: $K_N(z, \bar{w}) = \overline{K_N(w, \bar{z})}$;

- Normalization: $\int K_N(z, \bar{z}) d^2z = N$;

- Projection property: $\int K_N(z_1, \bar{z}) K_N(z, \bar{z}_2) d^2z = K_N(z_1, \bar{z}_2)$.

All density correlation functions can be expressed through the kernel function. For example:

$$\langle \rho(z) \rangle_N = K_N(z, \bar{z})$$

$$\langle \rho(z_1)\rho(z_2) \rangle_N = \begin{vmatrix} K_N(z_1, \bar{z}_1) & K_N(z_1, \bar{z}_2) \\ K_N(z_2, \bar{z}_1) & K_N(z_2, \bar{z}_2) \end{vmatrix} + K_N(z_1, \bar{z}_1)\delta(z_1 - z_2)$$

The last term is a *contact term.* It does not contribute if $z_1 \neq z_2$. In general, one has:

$$\langle \rho(z_1)\ldots\rho(z_n) \rangle_N = \det(K_N(z_i, \bar{z}_j))_{1 \leq i,j \leq n} + \text{contact terms}$$

where the contact terms vanish if all the points z_i are different.

Note that for the ensemble $\mathcal{N}$ with an axially-symmetric potential $W(\Phi) = W_0(\Phi\Phi^\dagger)$ the orthogonal polynomials are simply $P_n(z) = z^n$ and h_n is given explicitly:

$$h_n = \int |z|^{2n} e^{W_0(|z|^2)} d^2z = 2\pi \int_0^\infty r^{2n+1} e^{W_0(r^2)} dr$$

The kernel function is

$$K_N(z, \bar{w}) = e^{\frac{1}{2}(W(z)+W(w))} \sum_{n=0}^{N-1} \frac{(z\bar{w})^n}{h_n}$$

For example, for the Gaussian model with $W(z) = -|z|^2$ the squared norms are $h_n = \pi n!$ and the mean value of density is given by

$$\langle \rho(z) \rangle_N = \frac{1}{\pi} e^{-|z|^2} \sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!}$$

The Lax representation. The orthogonal polynomials obey the recurrence relation of the form $zP_n(z) = \sum_{k \leq n} c_{nk} P_k(z)$. In terms of the ψ-function it reads

$$z\psi_n(z) = r_n \psi_{n+1}(z) + \sum_{k \geq 0} u_k(n)\psi_{n-k}(z)$$

One can represent it as a "spectral problem" $L\psi = z\psi$ for the difference operator

$$L = r_n e^{\partial/\partial n} + \sum_{k \geq 0} u_k(n) e^{-k\partial/\partial n} \tag{3.17}$$

(the Lax operator). Here $e^{\partial/\partial n}$ is the shift operator $n \to n+1$ with the characteristic property $e^{\partial/\partial n} f(n) = f(n+1) e^{\partial/\partial n}$.

Let

$$W(z) = W^{(0)}(z) + \sum_k (t_k z^k + \bar{t}_k \bar{z}^k)$$

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It can be shown that the dependence on the parameters t_k, $\bar{t}_k$ is given by the *2D Toda hierarchy*

$$\frac{\partial}{\partial t_k}L = [A_k,\, L]\,, \qquad \frac{\partial}{\partial \bar{t}_k}L = [L,\, \bar{A}_k]$$

where $A_k = (L^k)_+ + \frac{1}{2}(L^k)_0$, $\bar{A}_k = (L^{\dagger k})_- + \frac{1}{2}(L^{\dagger k})_0$ and $L^\dagger = e^{-\partial/\partial n}r_n + \sum_{k\geq 0}e^{k\partial/\partial n}\bar{u}_k(n)$ is the conjugate Lax operator. Given an operator of the form $\hat{O} = \sum_k b_k e^{k\partial/\partial n}$, we use the standard definition $(\hat{O})_+ = \sum_{k>0}b_k e^{k\partial/\partial n}$, $(\hat{O})_- = \sum_{k<0}b_k e^{k\partial/\partial n}$ and $(\hat{O})_0 = b_0$. The structure of the Toda hierarchy in models of random matrices was first revealed in [27], see also review [4].

In the case of quasiharmonic potential, it is convenient to modify the ψ-functions:

$$\psi_n \to \chi_n = \frac{1}{\sqrt{h_{n-1}}}e^{V(z)}P_{n-1}(z)$$

The functions χ_n obey the orthogonality condition: $\int \chi_n(z)\overline{\chi_m(z)}e^{-|z|^2}d^2z = \delta_{nm}$. Then we have two compatible linear problems:

$$(L\chi)_n = z\chi_n\,, \qquad (L^\dagger \chi)_n = \partial_z \chi_n$$

The second equation can be proven by comparing the matrix elements of the both sides using integration by parts.

3.3   The loop equation

In matrix models, loop equations are exact relations which follow from the fact that the matrix integral defining the model does not depend on changes of integration variables. In our case, we may start directly from the integral over eigenvalues (2.7), thereby extending the result to any value of β.

Clearly, the integral (2.7) remains the same if we change the integration variables $z_i \to \tilde{z}_i$. In other words, it is invariant under reparametrizations of the z-coordinate, which we write in the infinitesimal form as $z_i \to z_i + \epsilon(z_i)$, $\bar{z}_i \to \bar{z}_i + \bar{\epsilon}(z_i)$. For the integral $Z_N = \int e^{-\beta E(z_1,\dots,z_N)}\prod_j d^2 z_j$ with E given in (2.10), the reparametrization yields, in the first order:

$$\prod_j d^2 z_j \longrightarrow \left[1 + \sum_l (\partial\epsilon(z_l) + \bar{\partial}\bar{\epsilon}(z_l))\right]\prod_j d^2 z_j$$

$$E \longrightarrow E + \sum_l \left(\frac{\partial E}{\partial z_l}\epsilon(z_i) + \frac{\partial E}{\partial \bar{z}_l}\bar{\epsilon}(z_i)\right)$$

The invariance of the integral is then expressed by the identity

$$\sum_i \int \frac{\partial}{\partial z_i}\left(\epsilon(z_i)e^{-\beta E}\right)\prod_j d^2 z_j = 0$$

valid for any ϵ. Introducing a suitable cutoff at infinity, if necessary, one sees that the 2D integral over z_i can be transformed, by virtue of the Green theorem, into a contour integral around infinity and so it does vanish.

Let us take $\epsilon(z_i) = \frac{1}{z - z_i}$, where z is a complex parameter. The singularity at the point z does not destroy the above identity since its contribution is proportional to the vanishing integral $\oint d\bar{z}_i/(z_i - z)$ over a small contour encircling z. Therefore, we have the equality

$$\sum_i \int \left[-\frac{\beta \partial_{z_i} E}{z - z_i} + \frac{1}{(z - z_i)^2} \right] e^{-\beta E} \prod_j d^2 z_j = 0$$

where $\partial_{z_i} E = -\sum_{l \neq i} \frac{1}{z_i - z_l} - \beta^{-1} \partial W(z_i)$ (see (2.10)). Using the identity

$$\sum_{i,j} \frac{1}{(z - z_i)(z - z_j)} = \sum_{i \neq j} \frac{2}{(z - z_i)(z_i - z_j)} + \sum_i \frac{1}{(z - z_i)^2}$$

we rewrite it in the form $\langle \mathcal{T}(z_1, \ldots, z_N) \rangle = 0$, where

$$\mathcal{T} = 2 \sum_i \frac{\partial W(z_i)}{z - z_i} + \beta \left(\sum_i \frac{1}{z - z_i} \right)^2 + (2 - \beta) \sum_i \frac{1}{(z - z_i)^2}$$

This identity gives an exact relation between one- and two-point correlation functions. To see this, we rewrite it in terms of $\varphi(z) = -\beta \sum_i \log |z - z_i|^2$ using the rule $\sum_i f(z_i) = \int f(z) \rho(z) d^2 z$. The result is *the loop equation*

$$\frac{1}{2\pi} \int \frac{\partial W(\zeta) \langle \Delta \varphi(\zeta) \rangle}{z - \zeta} d^2 \zeta = \langle T(z) \rangle \tag{3.18}$$

where

$$T(z) = (\partial \varphi(z))^2 + (2 - \beta) \partial^2 \varphi(z) \tag{3.19}$$

The correlator at coinciding points is understood as $\langle (\partial \varphi(z))^2 \rangle = \lim_{z' \to z} \langle \partial \varphi(z) \, \partial \varphi(z') \rangle$.

We have got an *exact* relation between one- and two-point correlation functions, valid for any finite N. For historical reasons, it is called the loop equation. One may read it as a Ward identity obeyed by correlation functions of the model. Being written in the form (3.18), (3.19), it resembles conformal Ward identities. Since correlation functions are variational derivatives of the free energy, the loop equation is an implicit functional relation for the free energy. However, it is not a closed relation. It can be made closed by some additional assumptions or approximations. A combination with $1/N$ expansion is particularly meaningful.

4 Large N limit

Starting from this section, we study the large N limit of the random matrix models introduced in Section 2. Our main tool is the loop equation. We shall see that in the large N limit meaningful geometric and algebro-geometric structures emerge, as well as important applications in physics.

4.1 Preliminaries

In order to be prepared for taking the large N ("quasiclassical") limit, it is convenient to introduce the "Planck constant" $\hbar$ by the rescaling $W(z) \to \frac{1}{\hbar}W(z)$, so the integral for the partition function acquires the form

$$Z_N = \int [D\Phi]e^{\frac{1}{\hbar}\operatorname{tr} W(\Phi)} \propto \int |\Delta_N|^{2\beta} \prod_j e^{\frac{1}{\hbar}W(z_j)}d^2z_j \tag{4.1}$$

which is ready for an $\hbar$-expansion.

Now we can specify what we mean by the large N limit. Namely, we are going to consider the integral (4.1) in the limit

$$N \to \infty, \quad \hbar \to 0, \quad \hbar N = t \quad \text{finite},$$

where t is a (positive) parameter having the dimension of area, or, equivalently, the integral

$$Z_N = \int |\Delta_N(z_i)|^{2\beta} \prod_j \exp\left(\frac{N}{t}W(z_j)\right) d^2z_j$$

as $N \to \infty$. With this convention, the $N \to \infty$ and $\hbar \to 0$ limits mean the same. It is natural to expect that $Z_N \xrightarrow{\hbar \to 0} e^{\hbar^{-2}F_0(t)}$, where the rescaled free energy F_0 is a smooth function of t (and parameters of the potential) finite as $N \to \infty$.

From now on we *change the normalization* of the functions ρ and φ (see (3.1), (3.3)) multiplying them by the $\hbar$:

$$\rho(z) = \hbar \sum_i \delta(z - z_i), \quad \varphi(z) = -\hbar\beta \sum_i \log|z - z_i|^2 \tag{4.2}$$

and use these definitions hereafter. (The idea is to make their mean values finite as $N \to \infty$.) The relation (3.4) between these functions remains unchanged. The density is now normalized as follows:

$$\int \rho(z)d^2z = t$$

Note that in our units the $\hbar$ has dimension of $[\text{length}]^2$, $\rho(z)$ is dimensionless and the partition function defined by (4.1) has dimension $[\text{length}]^{N(\beta N + 2 - \beta)}$, i.e., the combination

$$\hbar^{-\frac{1}{2}\beta N^2 + \frac{1}{2}(\beta - 2)N} Z_N = \int \left|\Delta_N\left(z_i/\sqrt{\hbar}\right)\right|^{2\beta} \prod_{j=1}^{N} \left(e^{\frac{1}{\hbar}W(z_j)}\frac{d^2z_j}{\hbar}\right) \tag{4.3}$$

is dimensionless.

In terms of the renormalized φ, the loop equation (3.18) has the form

$$\frac{1}{2\pi}\int \frac{\partial W(\zeta)\langle\Delta\varphi(\zeta)\rangle}{\zeta - z}d^2\zeta + \left\langle(\partial\varphi(z))^2\right\rangle + \varepsilon\left\langle\partial^2\varphi(z)\right\rangle = 0 \tag{4.4}$$

where

$$\varepsilon = (2 - \beta)\hbar \tag{4.5}$$

is small as $\hbar \to 0$. Note that ε is exactly zero for the ensemble $\mathcal{N}^0$ ($\beta = 2$), and so the last term does not enter the loop equation in this case. It is convenient to treat ε and $\hbar$ as independent small parameters.

4.2 Solution to the loop equation in the leading order

It is instructive to think about the large N limit under consideration in terms of the Dyson gas picture. Then the limit we are interested in corresponds to a very low temperature of the gas, when fluctuations around equilibrium positions of the charges are negligible. The main contribution to the partition function then comes from a configuration, where the charges are "frozen" at their equilibrium positions. It is also important that the temperature tends to zero simultaneously with increasing the number of charges, so the plasma can be regarded as a continuous fluid at static equilibrium. In the noninteracting fermions picture, this limit has some features of the quasiclassical approximation. Mathematically, all this means that the integral is evaluated by the saddle point method, with only the leading contribution being taken into account. As $\hbar \to 0$, correlation functions take their "classical" values $\langle \varphi(z) \rangle = \varphi_{cl}(z)$, and multipoint correlators factorize in the leading order: $\langle \partial \varphi(z) \, \partial \varphi(z') \rangle = \partial \varphi_{cl}(z) \partial \varphi_{cl}(z')$. Then the loop equation (4.4) becomes a *closed* relation for φ_{cl}:

$$\frac{1}{2\pi} \int \frac{\partial W(\zeta) \Delta \varphi_{cl}(\zeta)}{\zeta - z} \, d^2\zeta \quad + \left(\partial \varphi_{cl}(z) \right)^2 + \varepsilon \, \partial^2 \varphi_{cl}(z) \; = \; 0 \qquad (4.6)$$

Note that we hold the last term which is apparently of the next order in $\hbar$. The role of this term will be discussed below.

The case $\beta = 2$ (the ensemble $\mathcal{N}^0$). We begin with the case $\beta = 2$, when the last term in the r.h.s. of (4.6) vanishes exactly. Let us apply $\bar{\partial}$ to both sides of the equation. This yields:

$$-\partial W(z) \Delta \varphi_{cl}(z) + \partial \varphi_{cl}(z) \Delta \varphi_{cl}(z) = 0$$

Since $\Delta \varphi_{cl}(z) \propto \rho_{cl}(z)$ (see (4.2)), we obtain

$$\rho_{cl}(z) \left[\partial \varphi_{cl}(z) - \partial W(z) \right] = 0 \qquad (4.7)$$

This equation should be solved with the additional constraints $\int \rho_{cl}(z) d^2z = t$ (normalization) and $\rho_{cl}(z) \geq 0$ (positivity). The equation tells us that either $\partial \varphi_{cl}(z) = \partial W(z)$ or $\rho_{cl}(z) = 0$. Applying $\bar{\partial}$, we get $\Delta \varphi_{cl}(z) = \Delta W(z)$. Since $\Delta \varphi = -4\pi \beta \rho$, this gives the solution for ρ_{cl}:

$$\rho_{cl}(z) = -\frac{\Delta W(z)}{4\pi \beta} \qquad \text{"in the bulk"} \qquad (4.8)$$

Here, "in the bulk" just means "in the region where $\rho_{cl} > 0$". As we shall see below, this result holds true, up to some details, for other values of β as well, so β is kept in this formula and in some formulas below.

The physical meaning of the equation $\partial \varphi_{cl}(z) = \partial W(z)$ is clear. It is just the condition that the charges are in equilibrium (the saddle point for the integral). Indeed, the equation states that the total force experienced by a charge at any point z, where $\rho_{cl} \neq 0$, is zero. The interaction with the other charges, $\partial \varphi_{cl}(z)$, is compensated by the force $\partial W(z)$ due to the external field.
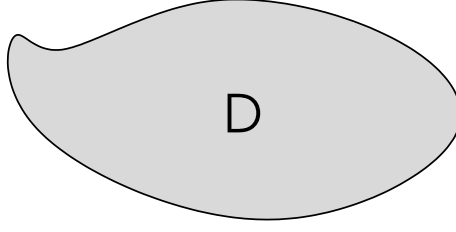
Figure 2: *The support of eigenvalues.*

Support of eigenvalues. Let us assume that

$$\sigma(z) := -\frac{1}{4\pi}\Delta W(z) > 0 \tag{4.9}$$

For quasiharmonic potentials, $\sigma(z) = 1/\pi$. If, according to (4.8), $\rho_{cl} = \sigma/\beta$ everywhere, the normalization condition for ρ_{cl} can not be satisfied! So we conclude that $\rho_{cl} = \sigma/\beta$ in a compact bounded domain (or domains) only, and outside this domain one should switch to the other solution of (4.7), $\rho_{cl} = 0$. The domain D where $\rho_{cl} > 0$ is called *support of eigenvalues* (Fig. 2). In general, it may consist of several disconnected components. The *complement* to the support of eigenvalues, $\mathsf{D}^{\mathsf{c}} = \mathbf{C} \setminus \mathsf{D}$, is an unbounded domain in the complex plane. For quasiharmonic potentials, the result is especially simple: ρ_{cl} is constant in D and 0 in D^{c}.

In terms of the mean value of the function $\varphi(z)$ in the leading order, the above result reads

$$\varphi_{cl}(z) = -\int_{\mathsf{D}} \log|z - \zeta|^2 \sigma(\zeta)\, d^2\zeta$$

As it follows from the theory of potential in two dimensions, this function is continuous across the boundary of D together with its first derivatives. However, the second order derivatives of this function have a jump across the boundary.

To find the shape of D is a much more challenging problem. It appears to be equivalent to the inverse potential problem in two dimensions. The shape of D is determined by the condition $\partial\varphi_{cl}(z) = \partial W(z)$ (imposed for all points z inside D) and by the normalization condition. Since $\partial\varphi_{cl}(z) = -\beta \int \frac{\rho_{cl}(\zeta)d^2\zeta}{z-\zeta}$ (see (4.2)), we write them in the form

$$\begin{cases} \dfrac{1}{4\pi} \displaystyle\int_{\mathsf{D}} \dfrac{\Delta W(\zeta)d^2\zeta}{z-\zeta} = \partial W(z) & \text{for all } z \in \mathsf{D} \\[4mm] \displaystyle\int_{\mathsf{D}} \sigma(\zeta)d^2\zeta = \beta t \end{cases}$$

The integral over D in the first equation can be transformed to a contour integral by means of the Cauchy formula (see Appendix B). As a result, one obtains:

$$\oint_{\partial\mathsf{D}} \frac{\partial W(\zeta)d\zeta}{z-\zeta} = 0 \qquad \text{for all } z \in \mathsf{D}. \tag{4.10}$$

This means that the domain D has the following property: the function $\partial W(z)$ on its boundary is the boundary value of an analytic function in its complement D^{c}.

We continue our analysis for the quasiharmonic case, where

$$\partial W(z) = -\bar{z} + V'(z), \quad \Delta W(z) = 4\partial\bar{\partial}W(z) = -4$$

The normalization than means that the area of D is equal to $\beta\pi t$. Assume that:

- $V(z) = \sum t_k z^k$ is regular in D (say a polynomial)

- $0 \in \mathsf{D}$ (it is always the case if $-W$ has a local minimum at 0)

- D is connected

Then the first equation in (4.10) acquires the form

$$\frac{1}{2\pi i} \oint_{\partial\mathsf{D}} \frac{\bar{\zeta}d\zeta}{\zeta - z} = V'(z) \qquad \text{for } z \in \mathsf{D}$$

Expanding it near $z = 0$, we get:

$$t_k = \frac{1}{2\pi i k} \oint_{\partial\mathsf{D}} \bar{\zeta}\zeta^{-k}d\zeta = -\frac{1}{\pi k} \int_{\mathsf{D}} \zeta^{-k}d^2\zeta \tag{4.11}$$

We see that the "coupling constants" t_k are *harmonic moments* of $\mathsf{D}^\mathsf{c} = \mathbf{C} \setminus \mathsf{D}$ and the area of D is $\pi\beta t$.

It is the subject of the inverse potential problem to reconstruct the domain from its area and harmonic moments. (In the case when the support of eigenvalues has several disconnected components, some additional conditions are required.) In general, the problem has many solutions. But it is known that *locally*, i.e., for a small enough change $t \to t + \delta t$, $t_k \to t_k + \delta t_k$ the solution is unique.

The support of eigenvalues: a fine structure ($\beta \neq 2$). Let us take a closer look at the support of eigenvalues, taking into account the so far ignored term in (4.6). Applying $\bar{\partial}$ to the both sides of (4.6) yields, instead of (4.7), the equation

$$\partial\varphi_{cl}(z) - \partial W(z) + \frac{\varepsilon}{2} \partial \log \Delta\varphi_{cl}(z) = 0$$

with the extra term proportional to ε (4.5). Acting by $\bar{\partial}$ once again, we obtain the equation for ρ_{cl}

$$-\frac{\varepsilon}{8\pi}\Delta \log \rho_{cl}(z) + \beta\rho_{cl}(z) = \sigma(z) \tag{4.12}$$

which looks like the Liouville equation in the "background" $\sigma(z)$. The first term seems to be negligible as $\hbar \to 0$. However, one should be careful since the small parameter stands in front of the term with the highest derivative. As a matter of fact, this term is negligible only if ρ_{cl} is not small! Indeed, in a region where $\rho_{cl} \sim e^{-N}$, the term $\varepsilon\Delta \log \rho_{cl}$ is of order 1 and thus plays the dominant role.

In fact the equation states that ρ_{cl} never vanishes exactly but can be exponentially small as $N \to \infty$. In the bulk, where the charges are distributed with nonzero density at $N = \infty$, equation (4.12) systematically generates corrections to the value $\sigma(z)/\beta$. If $\sigma \neq \text{const}$, there are power-like corrections in ε, as well as exponentially small ones. We

conclude that the effect of the extra term is negligible everywhere except the very vicinity of the edge of the support of eigenvalues. Therefore, the result for ρ_{cl} can still be written in the form

$$\rho_{cl}(z) = \beta^{-1}\sigma(z)\,\Theta(z;\mathsf{D}) \tag{4.13}$$

where $\Theta(z;\mathsf{D})$ is the characteristic function of the domain D (which is 1 in D and 0 in $\mathsf{D^c}$). The role of the term $\varepsilon\Delta\log\rho_{cl}$ is to make the edge smooth. Around the edge, the density rapidly (but smoothly) drops down to zero over distances of order $\sqrt{\varepsilon}$. So, the boundary has got a "fine structure".

All this is in agreement with the form of the first nonvanishing correction to the mean density found from the loop equation (4.4). Let us write $\langle\varphi(z)\rangle = \varphi_{cl}(z) + \varphi_\hbar(z)$, where $\varphi_\hbar$ is of order $\hbar$. The result for the $\varphi_\hbar$ can be compactly written in terms of the function

$$\chi(z) = \log\sqrt{\pi\sigma(z)} \tag{4.14}$$

and its harmonic continuation $\chi^H(z)$ from the boundary of D to its exterior. Specifically, $\chi^H(z)$ is a harmonic function in $\mathsf{D^c}$ (regular at ∞) such that $\chi^H(z) = \chi(z)$ on the boundary. In other words, it is the solution of the (exterior) Dirichlet boundary value problem (see below). The loop equation yields

$$\varphi_\hbar(z) = \begin{cases} -\varepsilon\left[\chi(z) - \chi^H(\infty) + \tfrac{1}{2}\right], & z \in \mathsf{D} \\[2mm] -\varepsilon\left[\chi^H(z) - \chi^H(\infty)\right], & z \in \mathsf{D^c} \end{cases} \tag{4.15}$$

(Note the discontinuity of this function across the boundary.) The corresponding correction to the mean density $\rho_\hbar = \langle\rho\rangle - \rho_{cl}$ is

$$\rho_\hbar(z) = \frac{\varepsilon}{4\pi\beta}\left(\Theta(z;\mathsf{D})\Delta\chi(z) - \delta(z;\partial\mathsf{D})\partial_n\left(\chi(z) - \chi^H(z)\right) - \frac{1}{2}\delta'(z;\partial\mathsf{D})\right) \tag{4.16}$$

where $\delta(z;\partial\mathsf{D})$ is the delta function with the support on the boundary and $\delta'(z;\partial\mathsf{D})$ is its normal derivative (see Appendix B). Here and below, ∂_n is the normal derivative at the boundary, with the normal vector being directed to the exterior of the domain D. The correction is so singular because the zeroth approximation (4.13) is singular by itself. The singular function $\rho_\hbar$ is to be understood as being integrated with any smooth test function. The first term in (4.16) is a correction to the bulk density. The second one is a correction to the shape of the support of eigenvalues (it describes a small displacement of the edge). The third term signifies the presence of a double layer of charges around the boundary. This just means that the boundary is smoothed out.

For the normal self-dual matrices ($\beta = 2$) the correction $\rho_\hbar$ vanishes. Certainly, this does not mean that the boundary is sharp. It becomes smooth if higher corrections in $\hbar$ (caused by fluctuations of the particles) are taken into account.

For the normal matrix model ($\beta = 1$) with quasiharmonic potential equation (4.12) reads

$$-\frac{\hbar}{8}\Delta\log\rho_{cl}(z) + \pi\rho_{cl}(z) = 1$$

The obvious solution is $\rho_{cl} = 1/\pi$. But it is not normalizable! One must look for another solution. The right solution differs from the constant by exponentially small terms in
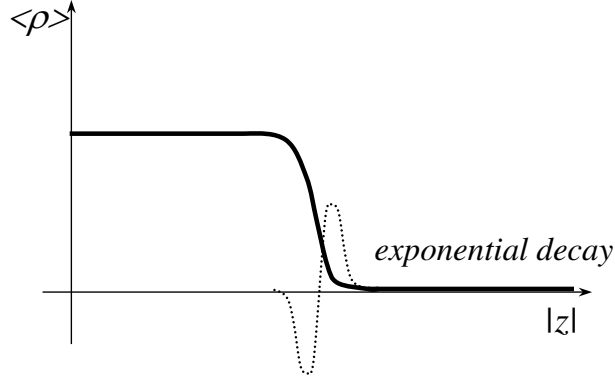
Figure 3: *The mean density profile for models with quasiharmonic potential in the large N limit. The correction to the pure step function is shown by the dotted line. The effect of the correction is to form a double layer of charges near the edge.*

the bulk but exhibits an abrupt drop across the boundary of the domain determined by the harmonic moments (4.11). So, up to exponentially small corrections, the solution is given by (4.13) with $\sigma = 1/\pi$ and $\beta = 1$. It is instructive to note that the first two terms in (4.16) vanish at $\sigma = \text{const}$ but the third term does not, so the leading correction in the quasiharmonic case merely makes the edge smooth that results in a double layer of "charges" near the edge (see Fig. 3).

From the support of eigenvalues to an algebraic curve. There is an interesting algebraic geometry behind the large N limit of matrix models. For simplicity, here we consider models with quasiharmonic potentials.

In general, the boundary of the support of eigenvalues is a closed curve in the plane without self-intersections. The following important fact holds true. If $V'(z)$ is a rational function, then this curve is a real section of a complex algebraic curve of finite genus. In fact, this curve encodes the $1/N$ expansion of the model. In the context of Hermitian 2-matrix model such a curve was introduced and studied in [28, 29, 30].

To explain how the curve comes into play, we start from the equation $\partial \varphi_{cl} = \partial W$, which can be written in the form $\bar{z} - V'(z) = G(z)$ for $z \in \mathsf{D}$, where

$$G(z) = \frac{1}{\pi} \int_{\mathsf{D}} \frac{d^2 \zeta}{z - \zeta}$$

Clearly, this function is analytic in D^{c}. At the same time, $V'(z)$ is analytic in D and all its singularities in D^{c} are poles. Set

$$S(z) = V'(z) + G(z)$$

Then $S(z) = \bar{z}$ on the boundary of the support of eigenvalues. So, $S(z)$ is the analytic continuation of $\bar{z}$ away from the boundary. Assuming that poles of V' are not too close to $\partial \mathsf{D}$, $S(z)$ is well-defined at least in a piece of D^{c} adjacent to the boundary. The complex

conjugation yields $\overline{S(z)} = z$, so the function $\bar{S}(z) = \overline{S(\bar{z})}$ must be inverse to the $S(z)$:

$$\bar{S}(S(z)) = z$$

("unitarity condition"). The function $S(z)$ is called the *Schwarz function* [31].

Under our assumptions, the $S(z)$ is an algebraic function, i.e., it obeys a polynomial equation $R(z, S(z)) = 0$ of the form

$$R(z, S(z)) = \sum_{n,l=1}^{d+1} a_{nl} z^n (S(z))^l = 0$$

where $\overline{a_{ln}} = a_{nl}$ and d is the number of poles of $V'(z)$ (counted with their multiplicities). Here is the sketch of proof. Consider the Riemann surface $\Sigma = \mathsf{D^c} \cup \partial \mathsf{D} \cup (\mathsf{D^c})^*$ (the *Schottky double* of $\mathsf{D^c}$). Here, $(\mathsf{D^c})^*$ is another copy of $\mathsf{D^c}$, with the local coordinate $\bar{z}$, attached to it along the boundary. On Σ, there exists an anti-holomorphic involution that interchanges the two copies of $\mathsf{D^c}$ leaving the points of $\partial \mathsf{D}$ fixed. The functions z and $S(z)$ are analytically extendable to $(\mathsf{D^c})^*$ as $\overline{S(z)}$ and $\bar{z}$ respectively. We have two meromorphic functions, each with $d+1$ poles, on a closed Riemann surface. Therefore, they are connected by a polynomial equation of degree $d+1$ in each variable. Hermiticity of the coefficients follows from the unitarity condition.

The polynomial equation $R(z, \tilde{z}) = 0$ defines a complex curve Γ with anti-holomorphic involution $(z, \tilde{z}) \mapsto (\bar{\tilde{z}}, \bar{z})$. The real section is the set of points such that $\tilde{z} = \bar{z}$. It is the boundary of the support of eigenvalues.

It is important to note that for models with non-Gaussian weights (in particular, with polynomial potentials of degree greater than two) the curve has a number of singular points, although the Riemann surface Σ (the Schottky double) is smooth. Generically, these are *double points*, i.e., the points where the curve crosses itself. In our case, a double point is a point $z^{(d)} \in \mathsf{D^c}$ such that $S(z^{(d)}) = \overline{z^{(d)}}$ but $z^{(d)}$ does not belong to the boundary of D. Indeed, this condition means that two different points on Σ, connected by the antiholomorphic involution, are stuck together on the curve Γ, which means the self-intersection. The double points play the key role in deriving the nonperturbative (instanton) corrections to the large N matrix models results (see [32] for details).

Finally, let us point out that a complex curve $\Gamma^{(n)}$ can be associated to ensembles of finite matrices as well (at least for $\beta = 1$). For the model of two Hermitian matrices this was done in [33]. If the linear spectral problem for the L-operator is of finite order (see the end of Section 3.2), the curve can be defined as the "spectral curve" of the difference spectral problems $(L\chi)_n = z\chi_n$, $(L^\dagger \chi)_n = \tilde{z}\chi_n$. Since the operators L and $L^\dagger$ do not commute, the curve depends on n. (See [8] for details.) In contrast to the curve Γ the curve $\Gamma^{(n)}$ is in general a smooth curve. A properly performed $n \to \infty$ limit of this curve coincides with the complex curve Γ constructed from the support of eigenvalues.

4.3 The free energy

In this subsection we assume that the support of eigenvalues is connected. As is known, the free energy admits a $1/N$ expansion. We prefer to work with the equivalent $\hbar$-

expansion, thus emphasizing its semiclassical nature. The first few terms of the $\hbar$-expansion for the ensembles of normal matrices with a general potential are

$$\log Z_N = c(N) + \frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + O(\hbar) \tag{4.17}$$

The explicit form of the $c(N)$ is given below. In fact this term can be absorbed into a normalization. For example, one may normalize Z_N dividing it by the partition function of the Gaussian model. Although in general the $\hbar$-expansion does not look like a topological one, it appears to be topological (i.e., only even powers of $\hbar$ enter) for the ensemble $\mathcal{N}^0$ with arbitrary potential and for the ensemble $\mathcal{N}$ with quasiharmonic potential.

The leading order. The partition function is given by (2.9), $Z_N = \int e^{-\beta E} \prod d^2 z_j$, where

$$-\beta E(z_1, \ldots, z_N) = \beta \sum_{i \neq j} \log |z_i - z_j| + \hbar^{-1} \sum_j W(z_j) \tag{4.18}$$

Writing the energy in terms of the density function, we have, in the leading order:

$$-\beta \hbar^2 E[\rho] = \beta \iint \rho(z)\rho(\zeta) \log |z - \zeta| d^2 z d^2 \zeta + \int W(z)\rho(z) d^2 z \tag{4.19}$$

We need to find the minimum of $E[\rho]$ with the constraint $\int \rho \, d^2 z = t$. This is achieved by variation of the functional $E[\rho] + \lambda(\int \rho \, d^2 z - t)$ with the Lagrange multiplier λ. The resulting equation is

$$2\beta \int \log |z - \zeta| \rho(\zeta) d^2 \zeta + W(z) + \lambda = 0$$

Upon taking the z-derivative, we see that the extremal $\rho(z)$ is equal to the $\rho_{cl}(z)$, as expected, and the equation coincides with the previously derived one, $\partial \varphi_{cl}(z) = \partial W(z)$, with

$$\varphi_{cl}(z) = -\beta \int \log |z - \zeta|^2 \rho_{cl}(\zeta) d^2 \zeta = -\int_{\mathsf{D}} \log |z - \zeta|^2 \sigma(\zeta) d^2 \zeta$$

Assuming that $W(0) = 0$ and D is connected, the Lagrange multiplier is fixed to be $\lambda = \varphi_{cl}(0)$, and so $W(z) = \varphi_{cl}(z) - \varphi_{cl}(0)$. Plugging this into (4.19), we find the leading contribution to the free energy $F_0/\hbar^2 = \max_\rho(-\beta E[\rho]) = -\beta E[\rho_{cl}]$:

$$F_0 = -\frac{1}{\beta} \iint_{\mathsf{D}} \int_{\mathsf{D}} \sigma(z) \log \left| \frac{1}{z} - \frac{1}{\zeta} \right| \sigma(\zeta) d^2 z d^2 \zeta \tag{4.20}$$

which is basically the electrostatic energy of the domain D charged with the density $\sigma(z)$ with a point-like compensating charge at the origin.

Since the t-derivative of the extremal value of the functional is equal to the Lagrange multiplier (with the sign minus), $\partial_t F_0 = -\lambda$, we incidentally obtain the useful formula

$$\partial_t F_0 = 2 \int_{\mathsf{D}} \log |z| \, \sigma(z) \, d^2 z \tag{4.21}$$

which will be rederived below by a more direct method.

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Corrections to the leading term. Taking into account the discrete "atomic" structure of the Dyson gas, one is able to find the subleading corrections to the free energy.

The first correction comes from a more accurate integral representation of the sum $\sum_{i \neq j} \log |z_i - z_j|$, when passing to the continuous theory. Namely, one should exclude the terms with $i = j$, writing

$$\sum_{i \neq j} \log |z_i - z_j| = \sum_{i,j} \log |z_i - z_j| - \sum_j \log |\ell(z_j)|$$

where ℓ is a short-distance cutoff (which may depend on the point z_j). It is natural to take the cutoff to be

$$\ell(z) \sim \sqrt{\frac{\hbar}{\rho_{cl}(z)}} \tag{4.22}$$

which is the mean distance between the charges around the point z. (In the context of the quantum Hall effect, $\ell \sim \sqrt{\hbar/B}$ is called the magnetic length.) This gives the improved estimate for $E[\rho_{cl}]$:

$$-\beta \hbar^2 E[\rho_{cl}] = F_0 + \beta \hbar \int \rho_{cl}(z) \sqrt{\rho_{cl}(z)} \, d^2 z - \frac{1}{2} \beta N \log \hbar + \alpha_1 N \tag{4.23}$$

where α_1 is a numerical constant which can not be determined by this argument.

Another correction comes from the integration measure when one passes from the integration over z_j to the integration over macroscopic densities[1]. We can write

$$\prod_j d^2 z_j = N! \, J[\rho] \, [D\rho]$$

where $[D\rho]$ is an integration measure in the space of densities, $J[\rho]$ is the Jacobian of this change of variables and the factor $N!$ takes into account the symmetry under permutations (all the states that differ by a permutation of the charges are identical). To estimate the Jacobian, we divide the plane into N microscopic "cells" such that j-th particle occupies a cell of size $\ell(z_j)$, where $\ell(z_j)$ is the mean distance (4.22) between the particles around the point z_j. All the microscopic states in which the particles remain in their cells are macroscopically indistinguishable. Given a macroscopic density ρ, $J[\rho]$ is then approximately equal to the integral $\int_{\text{cells}} \prod_j d^2 z_j$, with each particle being confined to its own cell. Therefore, $J[\rho] \sim \prod_j \ell^2(z_j)$, and thus $\log J[\rho]$ (sometimes referred to as entropy of the state with the macroscopic density ρ) is given by

$$\log J[\rho] = -\frac{1}{\hbar} \int \rho_{cl}(z) \log \rho_{cl}(z) \, d^2 z \; + \; N \log \hbar + \alpha_2 N \tag{4.24}$$

where α_2 is a numerical constant. This result agrees with the corresponding Jacobian obtained within the collective field theory approach [34].

Combining (4.23), (4.24) with $\rho = \rho_{cl}$, and taking into account the factor $N!$ in the measure, we obtain:

$$c(N) = \log N! + \frac{N}{2}(2 - \beta) \log \hbar + \alpha N \tag{4.25}$$

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[1] I thank A.Abanov for a discussion on this point.

where α is a numerical constant, and

$$F_{1/2} = -\frac{2-\beta}{2\beta} \int_{\mathsf{D}} \sigma(z) \log(\pi\sigma(z))\, d^2z \tag{4.26}$$

The term $F_{1/2}$ is thus the sum of the contribution due to the short-distance cutoff and the entropy contribution, which cancel each other in the ensemble of normal self-dual matrices (at $\beta = 2$). Another remarkable case when $F_{1/2}$ vanishes exactly[2] is the case of quasiharmonic potentials.

The result for $F_{1/2}$ can be derived in a more rigorous way from the loop equation (see Appendix C). Here we simply note that the variation of (4.26) over the potential W does yield the correction to the mean density given by (4.16). One can verify this using the variational technique presented below in Section 4.4.

The result for $c(N)$ is in agreement with the dimensionality argument. It is easy to see that $e^{F_0/\hbar^2}$ carries the dimension of $[\text{length}]^{\beta N^2}$, and higher terms are dimensionless. The dimension of Z_N is given by (4.3). Therefore, $e^{c(N)}$ must carry the residual dimension $[\text{length}]^{N(2-\beta)}$, which agrees with (4.25). For quasiharmonic potentials, the constant α can be found explicitly: $\alpha = \log\sqrt{2\pi^3}$, as is readily seen from the Gaussian case.

To summarize, the asymptotic expansion of the partition function as $\hbar \to 0$ has the form

$$Z_N = N!\,\hbar^{\frac{1}{2}(2-\beta)N} e^{\alpha N} \exp\left(\frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + \sum_{k\geq 3} \hbar^{k-2} F_{k/2}\right) \tag{4.27}$$

where F_0 and $F_{1/2}$ are given by (4.20) and (4.26) respectively. The higher corrections are due to fluctuations of the eigenvalues around the equilibrium configuration. No simple method to find their explicit form is known. In principle, these corrections can be found by expanding the loop equation in powers of $\hbar$ (see Appendix C), similarly to how it goes for the model of one Hermitian matrix [35, 36]. However, the calculations are rather tedious, even in the first two orders. At present only fragmentary results are available. Some of them look quite suggestive.

For example, the result for the F_1-correction obtained in [37] for the case of the normal matrix model with quasiharmonic potential and a connected support of eigenvalues has a clean interpretation as the free energy of the theory of free bosons in the domain D^{c} with the Dirichlet boundary conditions. Namely, the calculations yield the result

$$F_1 = -\frac{1}{24\pi} \oint_{|w|=1} \left(\log|z'(w)|\partial_n \log|z'(w)| + 2\log|z'(w)|\right)|dw| \tag{4.28}$$

where $z(w)$ is the univalent conformal map from D^{c} onto the exterior of the unit circle. Comparison with the Polyakov-Alvarez formula [38, 39] allows one to identify this quantity with $-\frac{1}{2}\log\det(-\Delta_{\mathsf{D}^{\mathsf{c}}})$, where $\det(-\Delta_{\mathsf{D}^{\mathsf{c}}})$ is the regularized determinant of the Laplace operator in D^{c} with the Dirichlet boundary conditions. This suggests the interpretation through free bosons[3]. Presumably, the higher corrections to the free energy are connected with spectral geometry of the Laplace operator, too. Recently, some progress in computation of F_1 and, more generally, in understanding the structure of the whole

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[2]However, its variational derivative, $\rho_\hbar$, does not vanish, as is seen from (4.16).

[3]A similar interpretation of the F_1-correction in the Hermitian matrix ensemble was suggested in [40].
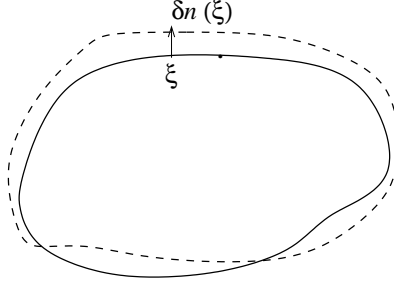
Figure 4: *The normal displacement of the boundary.*

series (including the case of disconnected supports) in models of Hermitian matrices was achieved [41]. Conjecturally, the answer is to be expressed in terms of a (conformal?) field theory on the complex curve Γ introduced at the end of Section 4.2.

Finally, we note that the structure of the loop equation suggests to rearrange the $\hbar$-expansion of the free energy and to write it in the "topological" form $F = \sum_{g \geq 0} \hbar^{2g} F_g$, where each term has its own expansion in $\varepsilon = (2 - \beta)\hbar$: $F_g = F_g^0 + \sum_{n \geq 1} \varepsilon^n F_g^{(n)}$.

4.4   Correlation functions in the large N limit

Variational technique and the Dirichlet boundary value problem.   Correlation functions in the leading order in $\hbar$ can be obtained from the free energy by variation w.r.t. $W(z)$ according to the formulas from Section 3.1. For a variation of the potential, $W \to W + \delta W$ with $N\hbar$ fixed we ask how D changes. It is convenient to describe small deformations $\mathsf{D} \to \tilde{\mathsf{D}}$, by the normal displacement $\delta n(\xi)$ of the boundary at a boundary point ξ (Fig. 4).

Consider a small variation of the potential W in the condition (4.10), which determines the shape of D at a fixed t. To take into account the deformation of the domain, $\delta \mathsf{D} = \tilde{\mathsf{D}} \setminus \mathsf{D}$, we write, for any fixed function f,

$$\delta \left(\oint_{\partial \mathsf{D}} f(\zeta)d\zeta \right) = \oint_{\partial(\delta \mathsf{D})} f(\zeta)d\zeta = 2i \int_{\delta \mathsf{D}} \bar{\partial} f(\zeta) d^2\zeta \approx 2i \oint_{\partial \mathsf{D}} \bar{\partial} f(\zeta)\delta n(\zeta)|d\zeta|$$

and thus obtain from (4.10):

$$\oint_{\partial \mathsf{D}} \frac{\partial\, \delta W(\zeta)d\zeta}{z - \zeta} + \frac{i}{2} \oint_{\partial \mathsf{D}} \frac{\Delta W(\zeta)\delta n(\zeta)}{z - \zeta} |d\zeta| = 0 \tag{4.29}$$

Here the first term comes from the variation of W and the second one comes from the change of D.

This is an integral equation for the $\delta n(\zeta)$. It can be solved in terms of the exterior Dirichlet boundary value problem. Given any smooth function $f(z)$, let $f^H(z)$ be its *harmonic continuation* (already introduced in Section 4.2) from the boundary of D to its exterior, i.e., the function such that it is harmonic in D^{c}, $\Delta f^H = 0$, and regular at ∞, and $f^H(z) = f(z)$ for all $z \in \partial \mathsf{D}$. The harmonic continuation is known to be unique. Explicitly, a harmonic function can be reconstructed from its boundary value by means

of the *Dirichlet formula*

$$f^H(z) = -\frac{1}{2\pi} \oint_{\partial \mathsf{D}} f(\xi) \partial_n G(z,\xi) |d\xi| \tag{4.30}$$

The main ingredient of this formula is $G(z,\xi)$, which is the Green function of the domain D^c:

$$\Delta_z G(z,\zeta) = 2\pi\delta(z-\zeta) \quad \text{in } \mathsf{D}^c, \quad G(z,\zeta) = 0 \quad \text{if } z \in \partial\mathsf{D}$$

As $\zeta \to z$, it has the logarithmic singularity $G(z,\zeta) \to \log|z-\zeta|$. The well known properties of harmonic functions imply that $\partial_n G(z,\xi) \le 0$ for all $\xi \in \partial\mathsf{D}$ (the operator of the normal derivative acts here to the second argument).

Consider the integral

$$\oint_{\partial\mathsf{D}} \frac{\partial(\delta W^H)d\zeta}{z-\zeta}$$

which is obviously equal to 0 for all z inside D, subtract it from the first term in (4.29) and rewrite the latter in the form

$$\oint_{\partial\mathsf{D}} \frac{\partial(\delta W - \delta W^H)d\zeta}{z-\zeta} = \frac{i}{2} \oint_{\partial\mathsf{D}} \frac{\partial_n(\delta W - \delta W^H)}{z-\zeta} |d\zeta|$$

Here the integral over $d\zeta$ is transformed to the integral over the line element $|d\zeta|$. The normal derivative is taken in the exterior of the boundary. After this simple transformation our condition acquires the form

$$\oint_{\partial\mathsf{D}} \frac{\delta n_1(\zeta) + \hat{R}(\zeta)}{z-\zeta} |d\zeta| = 0 \qquad \text{for all } z \in \mathsf{D} \tag{4.31}$$

where $\delta n_1(z) := \Delta W(z)\delta n(z)$ (just for brevity) and $\hat{R}$ is the Neumann jump operator. Acting on a smooth function f, this operator gives the difference between the normal derivative of this function and the normal derivative of its harmonic extension:

$$\hat{R}f(z) = \partial_n^-(f(z) - f^H(z)) \tag{4.32}$$

The superscript indicates that the derivative is taken in the exterior of the boundary.

By properties of Cauchy integrals, it follows from (4.31) that $\left[\delta n_1(z) + \hat{R}\delta W(z)\right]\frac{|dz|}{dz}$ is the boundary value of an analytic function $h(z)$ in D^c such that $h(\infty) = 0$. For $z \in \mathsf{D}^c$, this function is given by

$$h(z) = \frac{1}{2\pi i} \oint_{\partial\mathsf{D}} \frac{\delta n_1(\zeta) + \hat{R}(\zeta)}{z-\zeta} |d\zeta|$$

If h is not identically zero, the number of zeros of g in D^c, counted with multiplicities, is given by the contour integral $\frac{1}{2\pi}\oint_{\partial\mathsf{D}} d(\arg h)$. Since $\arg h(z) = \arg(|dz|/dz) = -\theta(z)$, where θ is the angle between the tangent vector and the real axis, this integral is equal to 1. We conclude that the function h has exactly one simple zero outside the domain D. It is just the zero at ∞.

On the other hand, the variation of the normalization condition yields, in a similar manner:

$$\oint_{\partial\mathsf{D}} (\delta n_1 + \partial_n \delta W)|d\zeta| = 0 \tag{4.33}$$

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This relation implies that the zero at ∞ is at least of the 2-nd order. Indeed, expanding the Cauchy integral around ∞,

$$2\pi i h(z) \rightarrow \frac{1}{z} \oint_{\partial \mathsf{D}} (\delta n_1 + \partial_n \delta W - \partial_n \delta W^H)|d\zeta| + O(z^{-2})$$

one concludes, using the Gauss law $\oint_{\partial \mathsf{D}} \partial_n \delta W^H |d\zeta| = -\int_{\mathsf{D}^c} \Delta \delta W^H d^2\zeta = 0$, that the coefficient in front of $1/z$ vanishes. We have got a contradiction.

Therefore, $h(z) \equiv 0$, and so $\delta n_1(z) + \hat{R}\delta W(z) = 0$. This gives the following result for the normal displacement of the boundary caused by small changes of the potential $W \rightarrow W + \delta W$:

$$\delta n(z) = \frac{\partial_n^- (\delta W^H(z) - \delta W(z))}{\Delta W(z)} \tag{4.34}$$

Some results for the correlation functions. In order to find the correlation functions of traces, we use the general variational formulas (3.5), where the exact free energy is replaced by the leading contribution (4.20):

$$\lim_{\hbar \to 0} \langle \rho(z) \rangle = \frac{\delta F_0}{\delta W(z)} = \rho_{cl}(z), \qquad \lim_{\hbar \to 0} \langle \rho(z_1)\rho(z_2) \rangle_c = \hbar^2 \frac{\delta \rho_{cl}(z_1)}{\delta W(z_2)}$$

Basically, these are linear response relations used in the Coulomb gas theory [42]. In this approximation, the eigenvalue plasma is represented as a continuous charged fluid, so the information about its discrete microscopic structure is lost. So, these formulas give "smoothed" correlation functions in the first non-vanishing order in $\hbar$. They are correct at distances much larger than the mean distance between the charges.

Here are the main results for the correlation functions obtained by the variational technique. For details of the derivation see [7] and Appendix C.

The leading contribution to the one-trace function was already found in Section 4.2. Here we present the general result including the first subleading correction which can be found by variation of (4.26):

$$\beta \langle \mathrm{tr} f(\Phi) \rangle = \frac{1}{\hbar} \int_{\mathsf{D}} \sigma(z) f(z)\, d^2z$$

$$+ \frac{2-\beta}{8\pi} \left[\int_{\mathsf{D}} (1+\log \sigma(z))\Delta f(z)\, d^2z - \oint_{\partial \mathsf{D}} \log \sigma(z)\, \hat{R}f(z)|dz| \right] + O(\hbar) \tag{4.35}$$

where $\hat{R}$ is the Neumann jump operator (4.32). Applying this formula to the function $\varphi(z)$, we get the familiar result $\langle \varphi(z) \rangle = \varphi_{cl}(z) + \varphi_\hbar(z) + O(\hbar^2)$, where $\varphi_\hbar$ is given by (4.15). The connected two-trace function is:

$$\beta \langle \mathrm{tr} f\, \mathrm{tr} g \rangle_c = \frac{1}{4\pi} \int_{\mathsf{D}} \nabla f \nabla g d^2z - \frac{1}{4\pi} \oint_{\partial \mathsf{D}} f \partial_n g^H |dz| + O(\hbar) \tag{4.36}$$

In particular, for the connected correlation functions of the fields $\varphi(z_1)$, $\varphi(z_2)$ (see (4.2)) this formula gives (if $z_{1,2} \in \mathsf{D}^c$):

$$\frac{1}{2\beta\hbar^2} \langle \varphi(z_1)\varphi(z_2) \rangle_c = G(z_1, z_2) - G(z_1, \infty) - G(\infty, z_2) - \log \frac{|z_1 - z_2|}{r} + O(\hbar) \tag{4.37}$$

where G is the Green function of the Dirichlet boundary value problem and

$$r = \exp\left[\lim_{\xi \to \infty}\left(\log|\xi| + G(\xi, \infty)\right)\right] \tag{4.38}$$

is the (external) conformal radius of the domain D. The 2-trace functions are *universal*, i.e., they depend on the shape of the support of eigenvalues only and do not depend on the potential W explicitly. They resemble the two-point functions of the Hermitian 2-matrix model found in [18]; they were also obtained in [43] in the study of thermal fluctuations of a confined 2D Coulomb gas. The structure of the formulas indicates that there are local correlations in the bulk as well as strong long range correlations at the edge of the support of eigenvalues. (See [44] for a similar result in the context of classical Coulomb systems).

From the mathematical point of view, the significance of formula (4.37) is to provide a link between such seemingly unrelated disciplines as classical analysis in two dimensions and the random matrix theory. Namely, different limits or certain specifications of the arguments in this formula allow one to represent some important objects of classical analysis associated with the domains D and D^{c} (e.g., the conformal map onto the unit circle and its Schwarzian derivative, the Bergman kernel) in terms of correlations between eigenvalues of random matrices.

Further variation of the pair density correlation function suggests that, starting from $n = 3$, the connected n-point density correlations vanish in the bulk in all orders of $\hbar$ (in fact they are exponential in $1/\hbar$). The entire leading contribution comes from the boundary. The result for the connected three-trace function is:

$$\beta\left\langle\prod_{i=1}^{3}\operatorname{tr} f_i\right\rangle_c = \frac{\hbar}{16\pi^2}\oint_{\partial\mathsf{D}}\frac{|dz|}{\sigma(z)}\prod_{j=1}^{3}\hat{R}f_j(z) + O(\hbar^2) \tag{4.39}$$

5 The matrix model as a growth problem

5.1 Growth of the support of eigenvalues

When N increases at a fixed potential W, one may say that the support of eigenvalues grows. More precisely, we are going to find how the shape of the support of eigenvalues changes under $t \to t + \delta t$, where $t = N\hbar$, if W stays fixed.

The starting point is the same as for the variations of the potential, and the calculations are very similar as well. Variation of the condition (4.10) and of the normalization condition yields

$$\oint_{\partial\mathsf{D}}\frac{\Delta W(\zeta)\delta n(\zeta)}{z - \zeta}|d\zeta| = 0 \quad \text{(for all } z \in \mathsf{D}\text{)}, \qquad \oint_{\partial\mathsf{D}}\Delta W(\zeta)\delta n(\zeta)\,|d\zeta| = -4\pi\beta\delta t$$

The first equation means that $\Delta W(z)\delta n(z)\frac{|dz|}{dz}$ is the boundary value of an analytic function $h(z)$ such that $h(z) = -4\pi\beta\delta t/z + O(z^{-2})$ as $z \to \infty$. The solution for the $\delta n(z)$ is:

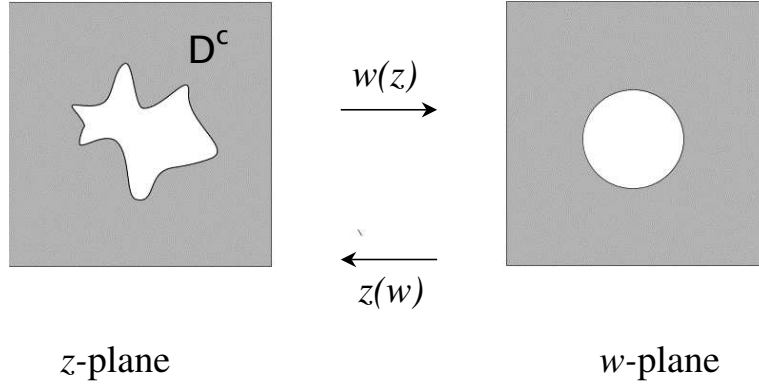$$\delta n(z) = -\frac{\beta\delta t}{2\pi\sigma(z)}\partial_n G(\infty, z) \tag{5.1}$$

Figure 5: *The mutually inverse conformal maps $w(z)$ and $z(w)$.*

where G is the Green function of the Dirichlet boundary problem in D^{c}. For quasiharmonic potentials (with $\sigma = 1/\pi$), the formula simplifies:

$$\delta n(z) = -\frac{\beta}{2}\,\delta t\,\partial_n G(\infty, z) \tag{5.2}$$

Identifying t with time, one can say that the normal velocity of the boundary, $V_n = \delta n/\delta t$, is proportional to gradient of the Green function: $V_n \propto -\partial_n G(\infty, z)$. This result is quite general. It holds for any (not necessarily connected) domains of eigenvalues with a smooth boundary.

If the domain is connected, the Green function can be expressed through the conformal map $w(z)$ from D^{c} onto the exterior of the unit circle:

$$G(z_1, z_2) = \log \left| \frac{w(z_1) - w(z_2)}{1 - w(z_1)\overline{w(z_2)}} \right| \tag{5.3}$$

In particular, $G(\infty, z) = -\log|w(z)|$. As $|z| \to \infty$, $w(z) = z/r + O(1)$, where r is the external conformal radius of the domain D which enters eq. (4.37). It is easy to see that $\partial_n \log|w(z)| = |w'(z)|$ on $\partial\mathsf{D}$, so one can rewrite the growth law (5.2) as follows:

$$\delta n(z) = \frac{\beta}{2}\delta t\,|w'(z)| \tag{5.4}$$

It is worth noting that the inverse map, $z(w)$, is the classical ($\hbar \to 0$) limit of the Lax operator (3.17) of the 2D Toda hierarchy. Indeed, making the rescaling $n\hbar = t$, $\partial/\partial n = \hbar\partial_t$, we see that the shift operator $e^{\hbar\partial_t}$ can be replaced by a commuting variable w with the Poisson bracket $\{\log w, t\} = 1$. In this limit, known also as the dispersionless limit of the 2D Toda hierarchy, the Lax operator $L(e^{\hbar\partial_t})$ converts into the function $z(w)$ given by the series of the form

$$z(w) = r(t)w + \sum_{k \geq 0} u_k(t)w^{-k} \tag{5.5}$$

It defines the one-to-one conformal map from the exterior of the unit circle onto D^{c}, which is inverse to the map $w(z)$ (Fig. 5). For more details see [8].

The basic formula which allows one to find the t-derivative of any quantity of the form $\int_{\mathsf{D}(t)} f(z)\, d^2z$ (where the function f is assumed to be independent of t) immediately follows from (5.1):

$$\delta\left(\int_{\mathsf{D}} f(z)\, d^2z\right) = \oint_{\mathsf{D}} f(z)\delta n(z)\,|dz| = -\frac{\beta\delta t}{2\pi}\oint_{\partial\mathsf{D}}\frac{f(z)}{\sigma(z)}\partial_n G(\infty, z)\,|dz|$$

In the r.h.s. we recognize the value at infinity of the harmonic continuation of the function $f(z)/\sigma(z)$, so the result is

$$\frac{\partial}{\partial t}\left(\int_{\mathsf{D}} f(z)\, d^2z\right) = \beta\,(f/\sigma)^H(\infty) \tag{5.6}$$

Using this formula and the integral representation of F_0, we find t-derivatives of the free energy. The first derivative,

$$\partial_t F_0 = 2\int_{\mathsf{D}}\sigma(z)\log|z|\, d^2z \tag{5.7}$$

was already found in Section 4.3 by other means (see (4.21)). The second derivative is proportional to the logarithm of the conformal radius (4.38):

$$\partial_t^2 F_0 = 2\beta\left.(\log|z|)^H\right|_{z=\infty} = 2\beta\lim_{z\to\infty}(\log|z| + G(z, \infty)) = 2\beta\log r \tag{5.8}$$

In this connection let us also mention the nice formula for the conformal map $w(z)$,

$$w(z) = \lim_{N\to\infty}\frac{\psi_{N+1}(z)}{\psi_N(z)} \tag{5.9}$$

which follows from the relation $\log\left(P_{N+1}(z)/P_N(z)\right) = \hbar\partial_t\langle\operatorname{tr}\log(z - \Phi)\rangle + O(\hbar)$ after calculating the t-derivative according to the above rule. This connection between conformal maps and orthogonal polynomials goes back to the classical theory of analytic functions (see e.g. [45]). It is the context of the random matrix theory where it looks natural and easily understandable.

5.2 Laplacian growth

The growth law (5.2) is common to many important problems in physics. The class of growth processes, in which dynamics of a moving front (an interface) between two distinct phases is driven by a harmonic scalar field is known under the name *Laplacian growth*. The most known examples are viscous flows in the Hele-Shaw cell (the Saffman-Taylor problem), filtration processes in porous media, electrodeposition and solidification of undercooled liquids. A comprehensive list of relevant papers published prior to 1998 can be found in [46]. Recently, the Laplacian growth mechanism was recognized [10] in a purely quantum evolution of semiclassical electronic blobs in the QH regime.
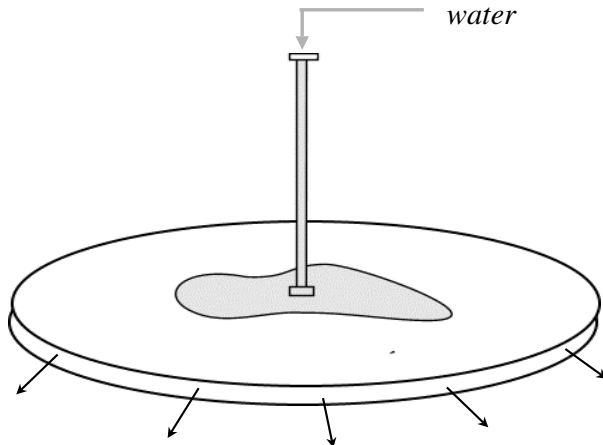
Figure 6: *The Hele-Shaw cell.*

The Saffman-Taylor problem. Let us describe the main features of the Laplacian growth on the example of viscous flows. To be specific, we shall speak about an interface between two incompressible fluids with very different viscosities on the plane (say, oil and water). In practice, the 2D geometry is realized in the Hele-Shaw cell – a narrow gap between two parallel glass plates. In this version, the problem is also known as the Saffman-Taylor problem or viscous fingering. For a review, see [14]. The velocity field in a viscous fluid in the Hele-Shaw cell is proportional to the gradient of pressure p (Darcy's law):

$$\vec{V} = -K\nabla p\,, \quad K = \frac{b^2}{12\mu}$$

Here the constant K is called the filtration coefficient, μ is viscosity and b is the size of the gap between the two plates. Note that if $\mu \to 0$, then $\nabla p \to 0$, i.e., pressure in a fluid with negligibly small viscosity is uniform. Incompressibility of the fluids ($\nabla \vec{V} = 0$) implies that the pressure field is harmonic: $\Delta p = 0$. By continuity, the velocity of the interface between the two fluids is proportional to the normal derivative of the pressure field on the boundary: $V_n = -K\partial_n p$.

To be definite, we assume that the Hele-Shaw cell contains a bounded droplet of water surrounded by an infinite "sea" of oil (another possible experimental set-up is an air bubble surrounded by water). Water is injected into the droplet while oil is withdrawn at infinity at a constant rate, as is shown schematically in Fig. 6. The latter means that the pressure field behaves as $p \propto -\log|z|$ at large distances. We also assume that the interface between oil and water is a smooth closed curve γ which depends on time. As it was mentioned above, if viscosity of water is negligible, then one may set $p = 0$ inside the water droplet. However, pressure usually has a jump across the interface, so p in general does not tend to zero if one approaches the boundary from outside. This effect is due to *surface tension*. It is hard to give realistic estimates of the surface tension effect from first principles, so one often employs certain ad hoc assumptions. The most popular one is to say that the pressure jump is proportional to the local curvature of the interface.

To summarize, the mathematical setting of the Saffman-Taylor problem is as follows:

$$\begin{cases} V_n = -\partial_n p & \text{on } \gamma \\ \Delta p = 0 & \text{in oil} \\ p \to -\log|z| & \text{in oil as } z \to \infty \\ p = 0 & \text{in water} \\ p^{(+)} - p^{(-)} = -\nu\kappa & \text{across } \gamma \end{cases} \tag{5.10}$$

Here ν is the surface tension coefficient and κ is the local curvature of the interface. (The filtration coefficient is set to be 1.) The experimental evidence suggests that when surface tension is small enough, the dynamics becomes unstable. Any initial domain develops an unstable fingering pattern. The fingers split into new ones, and after a long lapse of time the water droplet attains a fractal-like structure. This phenomenon is similar to the formation of fractal patterns in the diffusion-limited aggregation.

Comparing (5.2) and (5.10), we identify the D and D^{c} with the domains occupied by water and oil respectively, and conclude that the growth laws are identical, with the pressure field being given by the Green function: $p(z) = G(\infty, z)$, and $p = 0$ on the interface. The latter means that supports of eigenvalues grow according to (5.10) with *zero surface tension*, i.e., with $\nu = 0$ in (5.10).

Neglecting the surface tension effects, one obtains a good approximation unless the curvature of the interface becomes large. We see that the idealized Laplacian growth problem, i.e., the one with zero surface tension, is mathematically equivalent to the growth of the support of eigenvalues in ensembles of random matrices $\mathcal{N}$, $\mathcal{N}^0$ and $\mathcal{C}$. This fact clarifies the origin of the integrable structure of the Laplacian growth with zero surface tension discovered in [47]. The link to the normal matrix model has been established in [6], see also [48].

The finite-time singularities. As a matter of fact, the Laplacian growth problem with zero surface tension is ill-posed since an initially smooth interface often becomes singular in the process of evolution, and the solution blows up. The role of surface tension is to inhibit a limitless increase of the interface curvature. In the absense of such a cutoff, the tip of the most rapidly growing finger typically grows to a singularity (a cusp). In particular, a singularity necessarily occurs for any initial interface that is the image of the unit circle under a rational conformal map, with the only exception of an ellipse.

An important fact is that the cusp-like singularity occurs at a finite time $t = t_c$, i.e., at a finite area of the droplet. It can be shown that the conformal radius of the droplet r (as well as some other geometric parameters), as $t \to t_c$, exhibits a singular behaviour

$$r - r_c \propto (t_c - t)^{-\gamma}$$

characterized by a critical exponent γ. The generic singularity is the cusp $(2,3)$, which in suitable local coordinates looks like $y^2 = x^3$. In this case $\gamma = -\frac{1}{2}$. The evolution can not be extended beyond t_c.

A similar phenomenon was well-known in the theory of random matrices for quite a long time, and in fact it was the key to their applications to 2D quantum gravity and string theory. In the large N limit, the random matrix models have *critical points* – the

points where the free energy is not analytic as a function of a coupling constant. As we have seen, the Laplacian growth time t should be identified with a coupling constant of the normal or complex matrix model. In a vicinity of a critical point,

$$F_0 \sim F_0^{\text{reg}} + \alpha(t_c - t)^{2-\gamma}$$

where the critical index γ (often denoted by γ_{str} in applications to string theory) depends on the type of the critical point. Accordingly, the singularities show up in correlation functions. Using the equivalence established above, we can say that the finite-time blow-up (a cusp-like singularity) of the Laplacian growth with zero surface tension is a *critical point* of the normal and complex matrix models.

5.3   The semiclassical limit for electrons in magnetic field

As we have seen in Section 2.3, the system of N electrons in the plane in non-uniform magnetic field, which fully occupy the lowest energy level, is equivalent to the ensemble of normal $N \times N$ matrices, where N is degeneracy of the level. In this section we study the QH droplet in the semiclassical regime. The equivalence with the large N limit of the matrix model suggests to identify the semiclassical QH droplet with the support of eigenvalues. Remarkably, it is this limit where one makes contact with the purely classical Saffman-Taylor problem.

A remark is in order. The limit $\hbar \to 0$ we are talking about is really a *semi*classical limit, or better to say "partially classical". Although the Planck constant $\hbar$ tends to zero, all the particles remain at the lowest (most quantum) energy level, assuming their mass is small or the magnetic field is large. The true quasiclassical limit would imply that the particles occupy higher energy levels.

We recall that the joint probability to find electrons at the points z_i is $|\Psi_N(z_1, \ldots, z_N)|^2$, with $\Psi_N \propto \det_{N \times N}[\psi_n(z_k)]$, where

$$\psi_n(z) = \frac{1}{\sqrt{h_{n-1}}} P_{n-1}(z) e^{W(z)/(2\hbar)} \tag{5.11}$$

are orthogonal one-particle wave functions for electrons in the magnetic field $B = -\frac{1}{2}\Delta W$ at the lowest level $E = 0$. The level is assumed to be completely filled, i.e., $n = 0, 1 \ldots, N = [\phi/\phi_0]$, where ϕ_0 is the flux quantum. Then the mean density of the electrons coincides with the expectation value of the density of eigenvalues in the normal or complex matrix model.

The degeneracy of the level cam be controlled in different ways. One of them is to assume the following arrangement (see Fig. 7). Let a strong uniform magnetic field $B_0 > 0$ be applied in a large disk of radius R_0. The disk is surrounded by a large annulus $R_0 < |z| < R_1$ with a magnetic field $B_1 < 0$ such that the total magnetic flux through the system is $N\phi_0$: $\pi B_0 R_0^2 - \pi B_1(R_1^2 - R_0^2) = N\phi_0$. The magnetic field outside the largest disk $|z| < R_1$ vanishes. The disk is connected through a tunnel barrier to a large capacitor that maintains a small positive chemical potential slightly above the zero energy. If the field B_0 is strong enough, the gap between the energy levels is large, and the higher levels can be neglected. In the case of the uniform fields B_0 and B_1 the QH droplet is a disk of radius $r_0 = \sqrt{N\hbar} \ll R_0$ trapped at the origin.
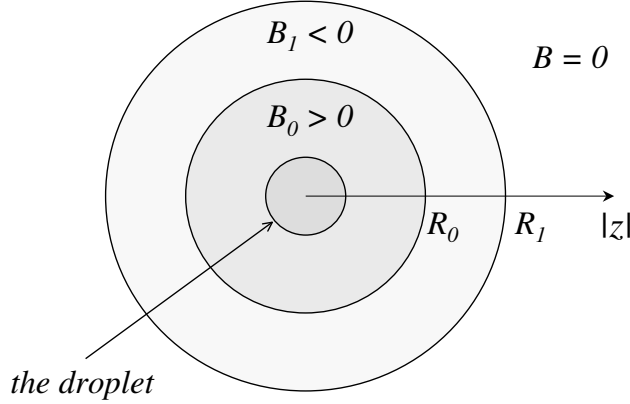
Figure 7: *The configuration of magnetic fields.*

In this set-up, let us apply a non-uniform magnetic field, δB, somewhere inside the disk $|z| < R_0$ but well away from the droplet. This leads to the following two effects.

The Aharonov-Bohm effect in the QH regime [10, 49]. Suppose that the nonuniform magnetic field δB does not change the total flux: $\int \delta B d^2 z = 0$. As is argued above, the shape of the droplet is the same as that of the support of eigenvalues in the ensemble $\mathcal{N}$ with the potential

$$W(z) = -\frac{B_0}{2}|z|^2 - \frac{1}{\pi} \int \log|z - \zeta|\, \delta B(\zeta)\, d^2\zeta$$

The second term is harmonic inside and around the droplet. One may have in mind thin solenoids carrying magnetic flux ("magnetic impurities"). In the case of point-like magnetic fluxes q_i at points a_i the change of the potential is $\delta W(z) = \sum_i q_i \log|z - a_i|$.

Let us stress that in the presence of the fluxes, the shape of the droplet is no longer circular although the magnetic field inside the droplet and not far from it remains uniform and is not changed at all (Fig. 8). In this respect this phenomenon is similar to the Aharonov-Bohm effect. Due to the quantum interference the electronic fluid is attracted to positive fluxes and is repelled by negative ones. The response of the droplet to an infinitesimal change of the magnetic field δB is described by eq. (4.34) in which

$$\delta W^H(z) - \delta W(z) = \frac{1}{\pi} \int_{\mathsf{D}^c} G(z, \zeta)\delta B(\zeta)d^2\zeta$$

In fact this formula holds for arbitrary δB, not necessarily vanishing inside the droplet. In particular, for small point-like fluxes δq_i at some points a_i we have $\delta W = \sum_i \delta q_i \log|z-a_i|$, $\delta B = -\pi \sum_i \delta q_i \delta^{(2)}(z - a_i)$, and $\delta W^H(z) - \delta W(z) = -\sum_i G(z, a_i)\delta q_i$. If a_i is inside, $G(z, a_i)$ is set to be zero. The sum, therefore, is over outside fluxes only. The fluxes inside the droplet, if any, appear to be completely screened and do not have any influence on its shape.
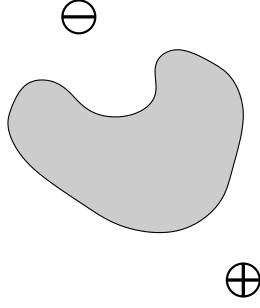
Figure 8: *The electronic droplet in the presence of magnetic impurities.*

Growth of the electronic droplet. If the total magnetic flux increases, with magnetic impurities kept fixed, the electronic droplet grows. For example, one may adiabatically increase B_1, with B_0 and δB fixed. Then the droplet grows because the degeneracy of the lowest level is enlarged and new electrons enter the system. The growth is described by eq. (5.1) with $\Delta W(z) = -2B_0$ which is equivalent to the Darcy law. This phenomenon is purely quantum. Like the Aharonov-Bohm effect, it is caused by quantum interference. Its characteristic scale is less than that of the Saffman-Taylor fingering by a factor of 10^9. The correspondence established above suggests that the edge of the QH droplet may develop unstable features similar to the fingers in the Hele-Shaw cell.

5.4   Semiclassical ψ-function

In this Subsection we derive the semiclassical asymptotics of the ψ-function (5.11), i.e.,

$$\psi_{N+1}(z) \xrightarrow{\hbar \to 0} \psi(z)$$

To avoid cumbersome technical details, we mainly consider models with quasiharmonic potential.

First of all, it is necessary to know the large N limit of the orthogonal polynomials P_N. Truncating the general formula (3.6) at the second term in the exponent, we can write:

$$P_N(z) = \langle \det(z - \Phi) \rangle = \left\langle e^{\operatorname{tr}\, \log(z-\Phi)} \right\rangle$$

$$\stackrel{\hbar \to 0}{=} \exp\left(\langle \operatorname{tr}\, \log(z - \Phi) \rangle + \frac{1}{2} \left\langle (\operatorname{tr}\, \log(z - \Phi))^2 \right\rangle_c + \ldots \right)$$

(5.12)

The first term in the r.h.s. is $O(\hbar^{-1})$, the second one is $O(\hbar^0)$ and the ignored terms vanish as $\hbar \to 0$. However, this formula should be applied with some care since the logarithm is not a single-valued function. This formula is correct if one can fix a single-valued branch of the logarithm. It is possible if z is outside D. For z inside D an analytic continuation should be used. To take care of the normalization, we also need

$$h_N = \frac{Z_{N+1}}{(N+1)Z_N} = (2\pi^3\hbar)^{1/2} \exp\left(\hbar^{-1}\partial_t F_0 + \frac{1}{2}\partial_t^2 F_0 + \ldots \right)$$

(5.13)

which is written here with the same precision as (5.12). (We have used (4.27) with $\alpha = \log\sqrt{2\pi^3\hbar}$ and have taken into account that $F_{1/2} = 0$ for quasiharmonic potentials.)

Let us first keep the dominant terms in the r.h.s. of (5.12), (5.13) and ignore the $O(\hbar^0)$ terms for a while. We have, for $z \in \mathsf{D^c}$: $|\psi(z)|^2 \sim e^{-\frac{1}{\hbar}\partial_t F_0} e^{\frac{1}{\hbar}(W(z) - \langle\varphi(z)\rangle)}$, or, using the results of the previous section,

$$|\psi(z)|^2 \sim e^{-2\mathcal{A}(z)/\hbar}, \qquad 2\mathcal{A}(z) = \varphi_{cl}(z) - \varphi_{cl}(0) - W(z)$$

As we know, $\mathcal{A}(z)$ defined by this formula is zero on the boundary. The analytic continuation of the $\mathcal{A}(z)$ inside the contour $\gamma = \partial\mathsf{D}$ can be done using the Schwarz function. Namely, since $2\partial\mathcal{A}(z) = \bar{z} - S(z)$, the desired analytic continuation can be defined (up to a constant) by the formula

$$2\mathcal{A}(z) = |z|^2 - 2\mathcal{R}e \int^z S(\zeta)\,d\zeta$$

Clearly, $\mathcal{A}(z)$ defined in this way is constant on the contour γ. Indeed, if $z_{1,2} \in \gamma$, then

$$2(\mathcal{A}(z_2) - \mathcal{A}(z_1)) = |z_2|^2 - |z_1|^2 - 2\mathcal{R}e \int_{z_1}^{z_2} S(z)dz$$

$$= z_2\bar{z}_2 - z_1\bar{z}_1 - \int_{z_1}^{z_2} \bar{z}dz - \int_{z_1}^{z_2} zd\bar{z} = z_2\bar{z}_2 - z_1\bar{z}_1 - \int_{z_1}^{z_2} d(z\bar{z}) = 0$$

Since $\mathcal{A}(z)$ should be zero on γ, we finally define

$$\mathcal{A}(z) = \frac{1}{2}|z|^2 - \frac{1}{2}|\xi_0|^2 - \mathcal{R}e \int_{\xi_0}^z S(\zeta)d\zeta \tag{5.14}$$

where ξ_0 is an arbitrary point on the contour. We call the function defined by (5.14) *the effective action*. From the above it follows that its first derivatives vanish for all $z \in \gamma$: $\partial\mathcal{A}(z) = \bar{\partial}\mathcal{A}(z) = 0$. This means that $|\psi|^2 \sim e^{-2\mathcal{A}/\hbar}$ has a sharp maximum on the contour γ. We may say that purely quantum particles are in general delocalized in the plane, purely classical particles are localized at some points in the plane while partially classical particles, like our electrons, are localized on closed curves in the plane.

Let us turn to the $O(\hbar^0)$-corrections, which give the subexponential factor in the asymptotics of the ψ-function[4]. We assume, for simplicity, that the domain D is connected. Extracting analytic and anti-analytic parts of eq. (4.37), we get

$$\left\langle (\mathrm{tr}\,\log(z - \Phi))^2 \right\rangle_c = \log(rw'(z)) + O(\hbar)$$

where r is the external conformal radius of D and $w'(z)$ is the derivative of the conformal map from $\mathsf{D^c}$ onto the exterior of the unit circle. Plugging this into (5.12) and taking into account eq. (5.8), we finally obtain:

$$|\psi(z)|^2 = \frac{|w'(z)|}{\sqrt{2\pi^3\hbar}} e^{-2\mathcal{A}(z)/\hbar} \tag{5.15}$$

This formula does resemble the WKB asymptotics in quantum mechanics. If a singularity of the conformal map is sufficiently close to the boundary from inside, the asymptotics becomes invalid in this region.

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[4]In what follows we restrict our consideration to the squared modulus $|\psi(z)|^2$. The ψ-function itself contains a phase factor which rapidly oscillates in both tangential and normal directions. (I thank A.Boyarsky and O.Ruchaisky for a discussion on this point.)

The effective action can be expanded near the contour, where it takes the minimal value:

$$\mathcal{A}(z + \delta_n z) = |\delta_n z|^2 \mp \frac{1}{3}\kappa(z)|\delta_n z|^3 + \frac{1}{4}\kappa^2(z)|\delta_n z|^4 + \ldots \tag{5.16}$$

Here $\kappa(z)$ is the local curvature of the contour at the point z and $\delta_n z$ is a small deviation from the point $z \in \gamma$ in the normal direction. (The upper and lower signs correspond to the outward and inward deviations respectively.) A similar expansion of $\log |w'(z)|$ reads

$$\log |w'(z + \delta_n z)| = \log |w'(z)| \pm (|w'(z)| - \kappa(z)) |\delta_n z| + \ldots \tag{5.17}$$

Some details of the derivation are given in Appendix E. Therefore, if $\kappa(z) \ll \hbar^{-1/2}$, the squared modulus of the ψ-function is well approximated by the sharp Gaussian distribution in the normal direction with the amplitude slowly modulated along the curve:

$$|\psi(z + \delta_n z)|^2 \simeq \frac{|w'(z)|}{\sqrt{2\pi^3 \hbar}} e^{-2|\delta_n z|^2/\hbar} \tag{5.18}$$

In the case of general potentials the calculations lead to a similar result:

$$|\psi(z + \delta_n z)|^2 \simeq \sqrt{\frac{\sigma(z)}{2\pi^2 \hbar}} |w'(z)| e^{-2\pi\sigma(z)|\delta_n z|^2/\hbar} \tag{5.19}$$

We see that the width of the Gaussian distribution depends on the point of the curve through the function σ (which is proportional to the magnetic field in the QH interpretation). Note that this asymptotics is consistent with the normalization $\int |\psi(z)|^2 d^2 z = 1$. The easiest way to see this is to notice that the Gaussian function in (5.18) (as well as the one in (5.19)) tends to the delta function $\delta(z; \gamma)$ with the support on the curve (see Appendix B). Hence one can formally write the limiting ψ-function in the form

$$|\psi(z)|^2 = \frac{|w'(z)|}{2\pi} \delta(z; \gamma) \tag{5.20}$$

and the right normalization is transparent.

Finally, we note that the growth law (5.4) (the Darcy law) can be written in the suggestive form

$$V_n(z) \propto |\psi(z)|^2 \tag{5.21}$$

The normal velocity V_n is defined by the relation

$$\partial_t \Theta(z; \mathsf{D}(t)) = V_n \delta(z; \partial \mathsf{D}(t))$$

where $\Theta(z; \mathsf{D})$ is the characteristic function of the domain D. Since the classical value of $\langle \rho(z) \rangle$ is $\Theta(z; \mathsf{D})/\pi$, we can represent the Darcy law in yet another form:

$$\partial_t \langle \rho(z) \rangle = |\psi(z)|^2$$

In fact it is the $\hbar \to 0$ limit of the exact relation

$$\langle \rho(z) \rangle_N - \langle \rho(z) \rangle_{N-1} = \hbar |\psi_N(z)|^2 \tag{5.22}$$

which immediately follows from the fact that $\langle \rho(z) \rangle_N = \hbar K_N(z, \bar{z})$ and from the definition of the kernel function (3.16). In fact it is equivalent to the Hirota equation (3.12). We see that (5.22) can be regarded as a "quantization" of the Darcy law.

Acknowledgments

Appendices

Appendix A

The proof of the determinant representation (3.7). It is simpler to start with $\det C_{ij}$, where C_{ij} is given in (3.8), and to show that $N! \det C_{ij}$ coincides with Z_N. The proof is a chain of obvious equalities. By definition,

$$N! \det C_{ij} = \sum_{Q,P} (-)^Q (-)^P \, C_{Q(1),P(1)} \ldots C_{Q(N),P(N)}$$

Plugging the explicit form of the C_{ij}, we write:

$$N! \det C_{ij} = \sum_{Q,P} (-)^Q (-)^P \int \prod_{i=1}^{N} \left[z_i^{Q(i)-1} \bar{z}_i^{P(i)-1} e^{W(z_i)} d^2 z_i \right]$$

and, interchanging the order of summation and integration, obtain the result:

$$N! \det C_{ij} = \int \underbrace{\left[\sum_Q (-)^Q \prod_{i=1}^{N} z_i^{Q(i)-1} \right]}_{\Delta_N(z_i)} \underbrace{\left[\sum_P (-)^P \prod_{i=1}^{N} \bar{z}_i^{P(i)-1} \right]}_{\overline{\Delta_N(z_i)}} \prod_{k=1}^{N} e^{W(z_k)} d^2 z_k \;=\; Z_N$$

Orthogonality of the polynomials (3.13). The definition (3.13) implies

$$P_n(\lambda) = \frac{1}{Z_n} \int |\Delta_n(z_i)|^2 \prod_{j=1}^{n} (\lambda - z_j) e^{W(z_j)} d^2 z_j$$

We want to show that these polynomials are orthogonal in the complex plane.

Set $e^{W(z)} d^2 z = d\mu$ for brevity. It is enough to show that $\int P_n(z) \bar{z}^m d\mu = 0$ for all $m < n$, i.e.,

$$\int d\mu(z) \bar{z}^m \int |\Delta_n(z_i)|^2 \prod_{j=1}^{n} (z - z_j) d\mu(z_j) = 0$$

Note that $\Delta_n(z_1, \ldots, z_n) \prod_{j=1}^{n} (z - z_j) = \Delta_{n+1}(z_1, \ldots, z_n, z)$. Setting $z \equiv z_{n+1}$, we have:

$$\text{LHS} = \int \Delta_{n+1}(z_i) \overline{\Delta_n(z_i)} \, \bar{z}_{n+1}^m \prod_{j=1}^{n+1} d\mu(z_j)$$

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$$= \frac{1}{n+1} \sum_{l=1}^{n+1} (-1)^{l+n+1} \int \Delta_{n+1}(z_1, \ldots, z_{n+1}) \overline{\Delta_n(z_1, \ldots, \not{z}_l, \ldots, z_{n+1}) z_l^m} \prod_{j=1}^{n+1} d\mu(z_j)$$

One can notice that the summation gives the expansion of the determinant

$$\begin{vmatrix} 1 & z_1 & \cdots & z_1^{n-1} & z_1^m \\ 1 & z_2 & \cdots & z_2^{n-1} & z_2^m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z_{n+1} & \cdots & z_{n+1}^{n-1} & z_{n+1}^m \end{vmatrix}$$

under the bar. If $m < n$, it vanishes. If $m = n$, it equals $\Delta_{n+1}(z_i)$, and we get LHS $= \frac{1}{n+1} Z_{n+1}$.

Appendix B

Here we list some standard formulas often used in Section 4.

The complex notation.

- Complex coordinates: $z = x + iy$, $\bar{z} = x - iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

- The Laplace operator: $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}$

- Contour integrals: let f and g be any smooth functions defined in some neighborhood of the contour γ, then

$$\oint_\gamma g\partial_n f |dz| = -2i \oint_\gamma g\partial_z f \, dz - i \oint f \, dg$$

Singular functions.

- Two-dimensional δ-function: $\delta(z) = \frac{1}{2\pi}\Delta \log |z| = \frac{1}{\pi}\partial_{\bar{z}}(1/z)$. The characteristic property of the delta-function is $f(z)\delta(z-a)\,d^2z = f(a)$ for any (smooth) function f.

- The δ-function with the support on a curve (a closed contour) γ: a function $\delta(z;\gamma)$ such that

$$\int f(z)\delta(z;\gamma)d^2z = \oint_\gamma f(z)|dz|$$

for any smooth function f.

- The "normal derivative" of the δ-function of the contour γ: a function $\delta'(z;\gamma)$ such that

$$\int f(z)\delta'(z;\gamma)d^2z = -\oint_\gamma \partial_n f(z)|dz|$$

for any smooth function f, with the normal vector being directed to the exterior of the contour.

- The characteristic function of the domain D: $\Theta(z;\mathsf{D}) = 1$ if $z \in \mathsf{D}$ and 0 otherwise; $\nabla\Theta(z;\mathsf{D}) = -\vec{n}\,\delta(z;\partial\mathsf{D})$.

Integral formulas.

- Cauchy's integral formula (f is any smooth function):

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(\zeta)d\zeta}{z-\zeta} - \frac{1}{\pi}\int_{D}\frac{\partial_{\bar\zeta}f(\zeta)d^2\zeta}{z-\zeta} = \begin{cases} -f(z), & z\in D \\ 0, & z\in C\setminus D \end{cases}$$

In particular, $\oint_{\partial D} f(\zeta)d\zeta = 2i\int_{D}\bar\partial f(\zeta)\,d^2\zeta$.

- The Green formula:

$$\int_{D} f\Delta g\, d^2z = -\int_{D}\nabla f\,\nabla g\,d^2z + \oint_{\partial D} f\partial_n g|dz|$$

where the normal vector looks outward D.

- The Dirichlet formula:

$$u(z) = -\frac{1}{2\pi}\oint_{\gamma} u(\zeta)\partial_n G(z,\zeta)|d\zeta|$$

for any function u harmonic in $D^c = C\setminus D$. Here $G(z,\zeta)$ is the Green function of the Dirichlet boundary value problem in D^c.

Appendix C

Let us present some details of the $\hbar$-expansion of the loop equation (4.4). First of all we rewrite it in the form

$$\frac{1}{2\pi}\int L(z,\zeta)\langle\Delta\varphi(\zeta)\rangle\, d^2\zeta = (\partial\varphi_{cl}(z))^2 - \left\langle(\partial(\varphi(z)-\varphi_{cl}(z))^2\right\rangle - (2-\beta)\hbar\left\langle\partial^2\varphi(z)\right\rangle$$

which is ready for the $\hbar$-expansion. Here

$$L(z,\zeta) = \frac{\partial W(\zeta) - \partial\varphi_{cl}(z)}{\zeta - z}$$

is the kernel of the integral operator in the l.h.s. (the "loop operator"). The zeroth order in $\hbar$ gives equation (4.7) which implies the familiar result $\varphi_{cl}(z) = -\int_{D}\log|z-\zeta|^2\sigma(\zeta)d^2\zeta$ for the φ_{cl}. To proceed, one should insert the series

$$\langle\varphi(z)\rangle = \varphi_{cl}(z) + \hbar\varphi_{1/2}(z) + \hbar^2\varphi_1(z) + O(\hbar^3)$$

(which corresponds to the $\hbar$-expansion (4.17) of the free energy) into the loop equation and separate terms of order $\hbar$, $\hbar^2$ etc. (In the notation adopted in the main body of the paper $\hbar\varphi_{1/2} = \varphi_\hbar + O(\hbar^2)$.) The terms of order $\hbar$ and $\hbar^2$ give:

$$\frac{1}{2\pi}\int L(z,\zeta)\langle\Delta\varphi_{1/2}(\zeta)\rangle\, d^2\zeta = -(2-\beta)\partial^2\varphi_{cl}(z)$$

$$\frac{1}{2\pi}\int L(z,\zeta)\langle\Delta\varphi_1(\zeta)\rangle\, d^2\zeta = -\left[\left(\partial\varphi_{1/2}(z)\right)^2 + (2-\beta)\partial^2\varphi_{1/2}(z)\right] - \omega(z)$$

where
$$\omega(z) = \lim_{\hbar \to 0} \left[\hbar^{-2} \lim_{z' \to z} \langle \partial\varphi(z)\,\varphi(z') \rangle_c \right]$$

is the connected part of the pair correlator at merging points. If the point z is in D^c, then eq. (4.37) yields

$$\langle \partial\varphi(z)\,\varphi(z') \rangle_c = 2\beta\hbar^2 \partial_z \partial_{z'} \Big(G(z, z') - \log|z - z'| \Big) + O(\hbar^3))$$

Since the r.h.s. is regular for all $z, z' \in \mathsf{D}^c$, the points can be merged without any regularization and the result does not depend on the particular limit $z' \to z$. We thus obtain that the function $\omega(z)$ is proportional to the Schwarzian derivative of the conformal map $w(z)$:

$$\omega(z) = \frac{\beta}{6} \left(\frac{w'''(z)}{w'(z)} - \frac{3}{2} \left(\frac{w''(z)}{w'(z)} \right)^2 \right)$$

The expansion of the loop equation can be continued order by order. In principle, this gives a recurrence procedure to determine the coefficients $\varphi_k(z)$. However, the equations of the chain are integral equations in the plane, and it is not easy to solve them explicitly. Another difficulty is that in general one can not extend these equations to the interior of the support of eigenvalues because the $\hbar$-expansion may break down or change its form there. Indeed, in the domain filled by the gas of eigenvalues the microscopic structure of the gas becomes essential, and one needs to know correlation functions at small scales. Nevertheless, at least in the first two orders in $\hbar^2$ the equations above can be solved assuming that $z \in \mathsf{D}^c$. Note that in this region all the functions $\varphi_k(z)$ are harmonic. If these functions are known, the corresponding expansion coefficients of the free energy in (4.17) can be obtained by "integration" of the variational formulas (3.5).

The procedure of solving the loop equation in the first two orders in $\hbar$ is too technical to be presented here. In the order $\hbar$ one is able to find a complete solution which gives formulas (4.15) and (4.26) mentioned in the main text. The solution in the next order, $\hbar^2$, is much more difficult to obtain. The results for φ_1 and F_1 are still not available in full generality (i.e., for general β and W). Nevertheless, for normal matrices with a general potential ($\beta = 1$) and with a connected support of eigenvalues the F_1-correction to the free energy can be found explicitly by the method outlined above. Here we present the result (mostly for an illustrative purpose), using the notation introduced in the main text (see (4.14), (5.5)):

$$
\begin{aligned}
F_1 &= -\frac{1}{24\pi} \oint_{|w|=1} \Big(\log|z'(w)| \partial_n \log|z'(w)| + 2\log|z'(w)| \Big) |dw| \\
&\quad - \frac{1}{24\pi} \left[\int_{\mathsf{D}} |\nabla\chi|^2\,d^2z + 2 \oint_{\partial\mathsf{D}} \kappa\chi\,|dz| \right] \\
&\quad + \frac{1}{8\pi} \left[\int_{\mathsf{D}} |\nabla\chi|^2\,d^2z - \oint_{\partial\mathsf{D}} \chi\partial_n\chi^H\,|dz| \right] - \frac{1}{16\pi} \int_{\mathsf{D}} \Delta\chi\,d^2z + c_0
\end{aligned}
$$

where c_0 is a numerical constant and $\kappa(z) = \partial_n \log \left| \frac{w(z)}{w'(z)} \right|$ is the local curvature of the boundary. For quasiharmonic potentials only the first integral survives.

Appendix D

Here we demonstrate how the variational technique works with the 2-trace correlator (4.36). We shall use the variational formulas (3.5) in the following equivalent version. Set $\delta W(z) = \epsilon g(z)$, where g is an arbitrary smooth function and $\epsilon \to 0$. Then, in the first order in ϵ,

$$\hbar \, \delta \, \langle \mathrm{tr} \, f \rangle = \epsilon \, \langle \mathrm{tr} \, f \, \mathrm{tr} \, g \rangle_c$$

This relation allows one to find the connected part of the two-trace correlation function by variation of the known one-trace function. Similar formulas hold for variations of multi-trace functions.

We have, in the leading order in $\hbar$:

$$\hbar \beta \delta \, \langle \mathrm{tr} \, f \rangle = \delta \left(\int_D \sigma f \, d^2 z \right) = \int_D \delta \sigma f \, d^2 z + \int_{\delta D} \sigma f \, d^2 z \equiv I_1 + I_2$$

The first integral, I_1, can be transformed using the definition of σ (4.9) and the Green formula:

$$I_1 = -\frac{1}{4\pi} \int_D \Delta(\delta W) f \, d^2 z = \frac{\epsilon}{4\pi} \int_D \nabla g \, \nabla f \, d^2 z - \frac{\epsilon}{4\pi} \oint_{\partial D} f \partial_n g \, |dz|$$

The second integral is

$$I_2 = -\frac{1}{4\pi} \int_D \Delta W(z) f(z) \delta n(z) |dz|$$

where $\delta n(z)$ is to be taken from eq. (4.34): $\delta n(z) = \epsilon \partial_n(g^H(z) - g(z))/\Delta W(z)$. Summing the two contributions, we get (4.36).

Appendix E

In this Appendix we obtain the expansions of the effective action $\mathcal{A}(z)$

$$\mathcal{A}(z) = \frac{1}{2}|z|^2 - \frac{1}{2}|\xi_0|^2 - \mathcal{R}e \int_{\xi_0}^{z} S(\zeta) \, d\zeta$$

near the contour $\gamma = \partial D$. We know that the first variation of $\mathcal{A}(z)$ vanishes on γ. To find the second variation, we write

$$2\delta \mathcal{A}(z) = |\delta z|^2 - \mathcal{R}e \left(S'(z)(\delta z)^2 \right), \quad z \in \gamma$$

Let us represent δz as a sum of normal and tangential deviations w.r.t. the curve: $\delta z = \delta_n z + \delta_t z$, then

$$2\mathcal{R}e \left(S'(z)(\delta z)^2 \right) = \mathcal{R}e \left(S'(z)(\delta_n z)^2 \right) + \mathcal{R}e \left(S'(z)(\delta_t z)^2 \right) + \mathcal{R}e \left(S'(z) \delta_n z \delta_t z \right)$$

Using the obvious relations

$$\frac{\delta_t z}{|\delta_t z|} = \sqrt{\frac{\delta_t z}{\delta_t \bar{z}}} = \frac{1}{\sqrt{S'(z)}}, \qquad \delta_n z = \mp i \left| \frac{\delta_n z}{\delta_t z} \right| \delta_t z$$

where the upper (lower) sign should be taken for the outward (inward) deviation, and the formula for the scalar product of 2-vectors $\vec{x}, \vec{y}$ represented as complex numbers x, y, $(\vec{x}, \vec{y}) = \mathcal{R}e\,(x\bar{y})$, we see that the first and second terms in the r.h.s. are equal to $-|\delta_n z|^2$ and $|\delta_t z|^2$ respectively while the third one vanishes since the vectors $\delta_n z$ and $\delta_t z$ are orthogonal. Since $|\delta z|^2 = |\delta_n z|^2 + |\delta_t z|^2$, we obtain the desired result $\delta \mathcal{A}(z) = |\delta_n z|^2$.

The next terms of the expansion of the $\mathcal{A}(z)$ around the contour can be found in a similar way. They are expressed through the curvature κ and its derivatives w.r.t. the arc length s along the curve. To perform the calculations in next two orders we need the following formulas for the κ and $\kappa' = d\kappa/ds$ through the Schwarz function [31]:

$$\kappa(z) = \frac{i}{2} \frac{S''(z)}{(S'(z))^{3/2}}, \qquad \kappa'(z) = \frac{i}{2} \frac{S'''(z)S'(z) - \frac{3}{2}(S''(z))^2}{(S'(z))^3}, \quad z \in \gamma$$

For z on the contour we have

$$\mathcal{A}(z + \delta_n z) = |\delta_n z|^2 - \frac{1}{6} \mathcal{R}e\,(S''(z)(\delta_n z)^3) - \frac{1}{24} \mathcal{R}e\,(S'''(z)(\delta_n z)^4) + \ldots$$

Now, with the help of the formulas for the curvature, it is easy to find that

$$\mathcal{R}e\,(S''(z)(\delta_n z)^3) = 2\kappa(z)\mathcal{R}e\left(\frac{1}{i}\left(\sqrt{S'(z)}\delta_n z\right)^3\right) = 2\kappa(z)\mathcal{R}e\left(\frac{1}{i}\left(\frac{\overline{\delta_t z}}{\delta_t z}\right)^{3/2}(\delta_n z)^3\right)$$

whence

$$\mathcal{R}e\,(S''(z)(\delta_n z)^3) = \pm 2\kappa(z)|\delta_n z|^3$$

A similar computation gives $\mathcal{R}e\,(S'''(z)(\delta_n z)^4) = -6\kappa^2(z)|\delta_n z|^4$ and we obtain the expansion (5.16). The expansion of $|w'(z)|$ around the contour can be easily performed with the help of the relations

$$\kappa(z) = \partial_n \log \left| \frac{w(z)}{w'(z)} \right|, \qquad \partial_n \log |w(z)| = |w'(z)|$$

valid for $z \in \gamma$.

References

[1] M.L.Mehta, *Random matrices and the statistical theory of energy levels*, 2-nd edition, Academic Press, NY, 1991

[2] T.Guhr, A.Müller-Groeling and H.Weidenmüller, Phys. Rep. **299** (1998) 189-428, e-print archive: cond-mat/9707301

[3] P.Di Francesco, P.Ginsparg and J.Zinn-Justin, Phys. Rep. 254 (1995) 1-133

[4] A.Morozov, Phys. Usp. **37** (1994) 1-55, e-print archive: hep-th/9303139

[5] Y.Fyodorov, B.Khoruzhenko and H.-J.Sommers, Phys. Rev. Lett. **79** (1997) 557, e-print archive: cond-mat/9703152; J.Feinberg and A.Zee, Nucl. Phys. **B504** (1997) 579-608, e-print archive: cond-mat/9703087; G.Akemann, J. Phys. A **36**

(2003) 3363, e-print archive: hep-th/0204246; G.Akemann, Phys. Rev. Lett. **89** (2002) 072002, e-print archive: hep-th/0204068 A.M.Garcia-Garcia, S.M.Nishigaki and J.J.M.Verbaarschot, Phys. Rev. **E66** (2002) 016132 e-print archive: cond-mat/0202151

[6] I.Kostov, I.Krichever, M.Mineev-Weinstein, P.Wiegmann and A.Zabrodin, *τ-function for analytic curves*, Random matrices and their applications, MSRI publications, eds. P.Bleher and A.Its, vol.40, p. 285-299, Cambridge Academic Press, 2001, e-print archive: hep-th/0005259

[7] P.Wiegmann and A.Zabrodin, J. Phys. A **36** (2003) 3411, e-print archive: hep-th/0210159

[8] R.Teodorescu, E.Bettelheim, O.Agam, A.Zabrodin and P.Wiegmann, Nucl. Phys. **B704/3** (2005) 407-444, e-print archive: hep-th/0401165; Nucl. Phys. **B700** (2004) 521-532, e-print archive: hep-th/0407017

[9] A.Marshakov, P.Wiegmann and A.Zabrodin, Commun. Math. Phys. **227** (2002) 131-153, e-print archive: hep-th/0109048

[10] O.Agam, E.Bettelheim, P.Wiegmann and A.Zabrodin, Phys. Rev. Lett. **88** (2002) 236801. e-print archive: cond-mat/0111333

[11] J.Ginibre, J. Math. Phys. **6** (1965) 440

[12] F.Dyson, J. Math. Phys. **3** (1962) 140, 157, 166

[13] R.B.Laughlin, in: The Quantum Hall Effect, R.E.Prange and S.M.Girvin (eds.), Springer, New York, 1987; S.Iso, D.Carabali and B.Sakita, Phys. Lett. **B296** (1992) 143; A.Cappelli, C.Trugenberger and G.Zemba, Nucl. Phys. **B396** (1993) 465

[14] D.Bensimon, L.P.Kadanoff, S.Liang, B.I.Shraiman, and C.Tang, Rev. Mod. Phys. **58** (1986) 977

[15] L.Chau and Y.Yu, Phys. Lett. **A167** (1992) 452-458; L.Chau and O.Zaboronsky, Commun. Math. Phys. **196** (1998) 203, e-print arhive: hep-th/9711091

[16] G.Oas, *Universal cubic eigenvalue repulsion for random normal matrices*, e-print archive: cond-mat/9610073

[17] M.Hastings, J. Stat. Phys **103** (2001) 903, e-print archive: cond-mat/9909234

[18] J.-M.Daul, V.Kazakov and I.Kostov, Nucl. Phys. **B409** (1993) 311

[19] S.Alexandrov, V.Kazakov and I.Kostov, Nucl. Phys. **B640** (2002) 119-144, e-print archive: hep-th/0205079; Nucl. Phys. **B667** (2003) 90-110, e-print archive: hep-th/0302106

[20] V.Girko, Theor. Prob. Appl. **29** (1985) 694

[21] P.Di Francesco, M.Gaudin, C.Itzykson and F.Lesage, Int. J. Mod. Phys. **A9** (1994) 4257-4351

[22] E.Marinari and G.Parisi, Phys. Lett. **B240** (1990) 375-380; E.Marinari and G.Parisi, Phys. Lett. **B247** (1990) 537-541

[23] M.Hastings, *Non-Hermitian fermion mapping for one-component plasma*, e-print archive: cond-mat/9703006

[24] A.Boyarsky, V.Cheianov and O.Ruchayskiy, *Fermions in the harmonic potential and string theory*, e-print archive: hep-th/0409129

[25] Y.Aharonov and A.Casher, Phys. Rev. **A19** (1979) 2461; J.Avron and R.Seiler, Phys. Rev. Lett. **42** (1979) 931

[26] R.Hirota, J. Phys. Soc. Japan, **50** (1981) 3785-3791

[27] A.Gerasimov, A.Marshakov, A.Mironov, A.Morozov and A.Orlov, Nucl. Phys. **B357** (1991) 565

[28] M.Staudacher, Phys. Lett **B305** (1993) 332-338, e-print archive: hep-th/9301038

[29] V.Kazakov and A.Marshakov, J. Phys. A **36** (2003) 3107, e-print archive: hep-th/0211236

[30] B.Eynard, JHEP 0301 (2003) 051, e-print archive: hep-th/0210047

[31] P.J.Davis, *The Schwarz function and its applications*, The Carus Math. Monographs, No. 17, The Math. Assotiation of America, Buffalo, N.Y., 1974

[32] F.David, Phys. Lett. **B302** (1993) 403-410, e-print archive: hep-th/9212106; B.Eynard and J.Zinn-Justin, Phys. Lett. **B302** (1993) 396-402, e-print archive: hep-th/9301004; V.Kazakov and I.Kostov, e-print archive: hep-th/0403152; S.Alexandrov, JHEP 0405 (2004) 025, e-print archive: hep-th/0403116

[33] M.Bertola, B.Eynard and J.Harnad Commun. Math. Phys. **229** (2002) 73-120, e-print archive: nlin.SI/0108049; Theor. Math. Phys. **134** (2003) 27-38, e-print archive: nlin.SI/0112006

[34] A.Jevicki and B.Sakita, Nucl. Phys. **B165** (1980) 511; A.Jevicki and B.Sakita, Nucl. Phys. **B185** (1981) 89; A.Jevicki, *Collective field theory and Schwinger-Dyson equations in matrix models*, preprint Brown-HET-777, Proceedings of the meeting "Symmetries, quarks and strings" held at the City College of New York, Oct. 1-2, 1990

[35] J.Ambjorn, L.Chekhov, C.F.Kristjansen and Yu.Makeenko, Nucl. Phys. **B404** (1993) 127-172; Erratum: ibid. **B449** (1995) 681, e-print archive: hep-th/9302014

[36] G.Akemann, Nucl. Phys. **B482** (1996) 403-430, e-print archive: hep-th/9606004; L.Chekhov, e-print archive: hep-th/0401089

[37] P.Wiegmann and A.Zabrodin, *Large N expansion for normal and complex matrix ensembles*, e-print archive: hep-th/0309253

[38] A.Polyakov, Phys. Lett. **B103** (1981) 207-210

[39] O.Alvarez, Nucl. Phys. **B216** (1983) 125-184; P.Di Vecchia, B.Durhuus, P.Olesen and J.Petersen, Nucl.Phys. **B207** (1982) 77; J.Ambjorn, B.Durhuus, J.Frölich and P.Orland, Nucl.Phys. **B270** (1986) 457

[40] I.Kostov, *Conformal Field Theory Techniques in Random Matrix models*, e-print archive: hep-th/9907060

[41] B.Eynard, A.Kokotov, D.Korotkin, Nucl. Phys. **B694** (2004) 443-472, e-print archive: hep-th/0403072; B.Eynard, JHEP 0411 (2004) 031, e-print archive: hep-th/0407261

[42] P.J.Forrester, Physics Reports, **301** (1998) 235-270; Log gases and random matrices webpage http://www.ms.unimelb.edu.au/ matjpf/matjpf.html

[43] A.Alastuey and B.Jancovici, J. Stat. Phys. **34** (1984) 557; B.Jancovici J. Stat. Phys. **80** (1995) 445

[44] B.Jancovici, J. Stat. Phys. **28** (1982) 43

[45] E.Hille, *Analytic function theory*, vol. 2, Needham Heights, MA: Ginn (1962)

[46] K.A.Gillow and S.D.Howison, *A bibliography of free and moving boundary problems for Hele-Shaw and Stokes flow*, http://www.maths.ox.ac.uk/ howison/Hele-Shaw/

[47] M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin, Phys. Rev. Lett. **84** (2000) 5106-5109, e-print archive: nlin.SI/0001007

[48] A.Zabrodin, Ann. Henri Poincaré **4** Suppl. 2 (2003) S851-S861, e-print archive: cond-mat/0210331

[49] P.Wiegmann, in: Statistical Field Theories, pp. 337-349, A. Cappelli and G. Mussardo (eds.), Kluwer Academic Publishers, 2002, e-print archive: cond-mat/0204254